(II) From (c) and (b), valid for $1 < |z| < 2$,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \frac{1}{2} + \frac{1}{4}z + \frac{1}{8}z^2 + \cdots - \frac{1}{z} - \frac{1}{z^2} - \cdots .$$

(III) From (d) and (b), valid for $|z| > 2$,

$$f(z) = -\sum_{n=0}^{\infty} (2^n + 1) \frac{1}{z^{n+1}} = -\frac{2}{z} - \frac{3}{z^2} - \frac{5}{z^3} - \frac{9}{z^4} - \cdots .$$

If $f(z)$ in Laurent's theorem is analytic inside C_2, the coefficients b_n in (2) are zero by Cauchy's integral theorem, so that the Laurent series reduces to a Taylor series. Examples 3(a) and 5(I) illustrate this.

PROBLEM SET 16.1

1–4   LAURENT SERIES NEAR A SINGULARITY AT 0

Expand the function in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence. Show the details of your work.

1. $\dfrac{\cos z}{z^4}$

2. $\dfrac{\exp(-1/z^2)}{z^2}$

3. $z^3 \cosh \dfrac{1}{z}$

4. $\dfrac{e^z}{z^2 - z^3}$

5–9   LAURENT SERIES NEAR A SINGULARITY AT z_0

Find the Laurent series that converges for $0 < |z - z_0| < R$ and determine the precise region of convergence. Show details.

5. $\dfrac{e^z}{(z-1)^2}$, $z_0 = 1$

6. $\dfrac{1}{z^2(z-i)}$, $z_0 = i$

7. $\dfrac{\sin z}{(z - \frac{1}{4}\pi)^3}$, $z_0 = \frac{1}{4}\pi$

8. CAS PROJECT. Partial Fractions. Write a program for obtaining Laurent series by the use of partial fractions. Using the program, verify the calculations in Example 5 of the text. Apply the program to two other functions of your choice.

9. TEAM PROJECT. Laurent Series. (a) Uniqueness. Prove that the Laurent expansion of a given analytic function in a given annulus is unique.

(b) Accumulation of singularities. Does $\tan(1/z)$ have a Laurent series that converges in a region $0 < |z| < R$? (Give a reason.)

(c) Integrals. Expand the following functions in a Laurent series that converges for $|z| > 0$:

$$\frac{1}{z^2} \int_0^z \frac{e^t - 1}{t} \, dt, \qquad \frac{1}{z^3} \int_0^z \frac{\sin t}{t} \, dt.$$

10–13   TAYLOR AND LAURENT SERIES

Find all Taylor and Laurent series with center z_0. Determine the precise regions of convergence. Show details.

10. $\dfrac{1}{1 - z^2}$, $z_0 = 0$

11. $\dfrac{1}{z}$, $z_0 = 1$

12. $\dfrac{1}{z^2}$, $z_0 = i$

13. $\dfrac{z^8}{1 - z^4}$, $z_0 = 0$

16.2 Singularities and Zeros. Infinity

Roughly, a *singular point* of an analytic function $f(z)$ is a z_0 at which $f(z)$ ceases to be analytic, and a *zero* is a z at which $f(z) = 0$. Precise definitions follow below. In this section we show that Laurent series can be used for classifying singularities and Taylor series for discussing zeros.

An analytic function whose only singularities in the finite plane are poles is called a **meromorphic function.** Examples are rational functions with nonconstant denominator, $\tan z$, $\cot z$, $\sec z$, and $\csc z$. ∎

In this section we used Laurent series for investigating singularities. In the next section we shall use these series for an elegant integration method.

PROBLEM SET 16.2

1–6   ZEROS

Determine the location and order of the zeros.

1. $\sin^4 \tfrac{1}{2} z$

2. $(z + 81i)^4$

3. $\tan^2 2z$

4. $\cosh^4 z$

5. **Zeros.** If $f(z)$ is analytic and has a zero of order n at $z = z_0$, show that $f^2(z)$ has a zero of order $2n$ at z_0.

6. **TEAM PROJECT. Zeros.** (a) **Derivative.** Show that if $f(z)$ has a zero of order $n > 1$ at $z = z_0$, then $f'(z)$ has a zero of order $n - 1$ at z_0.

   (b) **Poles and zeros.** Prove Theorem 4.

   (c) **Isolated k-points.** Show that the points at which a nonconstant analytic function $f(z)$ has a given value k are isolated.

   (d) **Identical functions.** If $f_1(z)$ and $f_2(z)$ are analytic in a domain D and equal at a sequence of points z_n in D that converges in D, show that $f_1(z) \equiv f_2(z)$ in D.

7–12   SINGULARITIES

Determine the location of the singularities, including those at infinity. For poles also state the order. Give reasons.

7. $\dfrac{1}{(z + 2i)^2} - \dfrac{z}{z - i} + \dfrac{z + 1}{(z - i)^2}$

8. $\tan \pi z$

9. $(z - \pi)^{-1} \sin z$

10. **Essential singularity.** Discuss e^{1/z^2} in a similar way as $e^{1/z}$ is discussed in Example 3 of the text.

11. **Poles.** Verify Theorem 1 for $f(z) = z^{-3} - z^{-1}$. Prove Theorem 1.

12. **Riemann sphere.** Assuming that we let the image of the x-axis be the meridians $0°$ and $180°$, describe and sketch (or graph) the images of the following regions on the Riemann sphere: (a) $|z| > 100$, (b) the lower half-plane, (c) $\tfrac{1}{2} \leqq |z| \leqq 2$.

16.3 Residue Integration Method

We now cover a second method of evaluating complex integrals. Recall that we solved complex integrals directly by Cauchy's integral formula in Sec. 14.3. In Chapter 15 we learned about power series and especially Taylor series. We generalized Taylor series to Laurent series (Sec. 16.1) and investigated singularities and zeroes of various functions (Sec. 16.2). Our hard work has paid off and we see how much of the theoretical groundwork comes together in evaluating complex integrals by the residue method.

The purpose of Cauchy's residue integration method is the evaluation of integrals

$$\oint_C f(z)\, dz$$

taken around a simple closed path C. The idea is as follows.

If $f(z)$ is analytic everywhere on C and inside C, such an integral is zero by Cauchy's integral theorem (Sec. 14.2), and we are done.

The situation changes if $f(z)$ has a singularity at a point $z = z_0$ inside C but is otherwise analytic on C and inside C as before. Then $f(z)$ has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

PROBLEM SET 16.3

1–5 RESIDUES

Find all the singularities in the finite plane and the corresponding residues. Show the details.

1. $\dfrac{\sin 2z}{z^6}$

2. $\dfrac{8}{1 + z^2}$

3. $\tan z$

4. $e^{1/(1-z)}$

5. **CAS PROJECT. Residue at a Pole.** Write a program for calculating the residue at a pole of any order in the finite plane. Use it for solving Probs. 5–6 (and online Probs. 7–10).

6–10 RESIDUE INTEGRATION

Evaluate (counterclockwise). Show the details.

6. $\displaystyle\oint_C \dfrac{z - 23}{z^2 - 4z - 5}\,dz, \quad C: |z - 2 - i| = 3.2$

7. $\displaystyle\oint_C \tan 2\pi z\,dz, \quad C: |z - 0.2| = 0.2$

8. $\displaystyle\oint_C e^{1/z}\,dz, \quad C: \text{the unit circle}$

9. $\displaystyle\oint_C \dfrac{\exp(-z^2)}{\sin 4z}\,dz, \quad C: |z| = 1.5$

10. $\displaystyle\oint_C \dfrac{z \cosh \pi z}{z^4 + 13z^2 + 36}\,dz, \quad |z| = \pi$

16.4 Residue Integration of Real Integrals

Surprisingly, residue integration can also be used to evaluate certain classes of complicated real integrals. This shows an advantage of complex analysis over real analysis or calculus.

Integrals of Rational Functions of $\cos\theta$ and $\sin\theta$

We first consider integrals of the type

(1)
$$ J = \int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta $$

where $F(\cos\theta, \sin\theta)$ is a real rational function of $\cos\theta$ and $\sin\theta$ [for example, $(\sin^2\theta)/(5 - 4\cos\theta)$] and is finite (does not become infinite) on the interval of integration. Setting $e^{i\theta} = z$, we obtain

(2)
$$ \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) $$
$$ \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right). $$

(clockwise!)

from $-\infty$ to
veral simple
ng residues.

ond over all

PROBLEM SET 16.4

1–4   INTEGRALS INVOLVING COSINE AND SINE

Evaluate the following integrals and show the details of
your work.

1. $\displaystyle\int_0^\pi \frac{2\,d\theta}{k - \cos\theta}$

2. $\displaystyle\int_0^{2\pi} \frac{1 + \sin\theta}{3 + \cos\theta}\,d\theta$

3. $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta}\,d\theta$

4. $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{13 - 12\cos 2\theta}\,d\theta$

5–8   IMPROPER INTEGRALS: INFINITE INTERVAL OF INTEGRATION

Evaluate the following integrals and show details of your
work.

5. $\displaystyle\int_{-\infty}^\infty \frac{dx}{(1 + x^2)^3}$

6. $\displaystyle\int_{-\infty}^\infty \frac{x^2 + 1}{x^4 + 1}\,dx$

7. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^4 - 1}$

8. $\displaystyle\int_{-\infty}^\infty \frac{x}{8 - x^3}\,dx$

9–14   IMPROPER INTEGRALS: POLES ON THE REAL AXIS

Find the Cauchy principal value (showing details):

9. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^4 - 1}$

10. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^4 + 3x^2 - 4}$

11. $\displaystyle\int_{-\infty}^\infty \frac{x + 5}{x^3 - x}\,dx$

12. $\displaystyle\int_{-\infty}^\infty \frac{x^2}{x^4 - 1}\,dx$

13. CAS EXPERIMENT. Simple Poles on the Real Axis. Experiment with integrals $\int_{-\infty}^\infty f(x)\,dx$, $f(x) = [(x - a_1)(x - a_2)\cdots(x - a_k)]^{-1}$, a_j real and all different, $k > 1$. Conjecture that the principal value of these integrals is 0. Try to prove this for a special k, say, $k = 3$. For general k.

14. TEAM PROJECT. Comments on Real Integrals.
  (a) **Formula (10)** follows from (9). Give the details.

  (b) **Use of auxiliary results.** Integrating e^{-z^2} around the boundary C of the rectangle with vertices $-a, a$, $a + ib, -a + ib$, letting $a \to \infty$, and using

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2},$$

  show that

$$\int_0^\infty e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}.$$

  (This integral is needed in heat conduction in Sec. 12.7.)

  (c) **Inspection.** Solve online Probs. 1 and 2 without calculation.

CHAPTER 16 REVIEW QUESTIONS AND PROBLEMS

1. What is a Laurent series? Its principal part? Its use? Give simple examples.

2. What kind of singularities did we discuss? Give definitions and examples.

3. What is the residue? Its role in integration? Explain methods to obtain it.

4. Can the residue at a singularity be zero? At a simple pole? Give reason.

5. State the residue theorem and the idea of its proof from memory.

6. How did we evaluate real integrals by residue integration? How did we obtain the closed paths needed?

7. What are improper integrals? Their principal value? Why did they occur in this chapter?

8. What do you know about zeros of analytic functions? Give examples.

9. What is the extended complex plane? The Riemann sphere R? Sketch $z = 1 + i$ on R.

10. What is an entire function? Can it be analytic at infinity? Explain the definitions.

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grals.