(b) We claim that the series in (5) can be written

$$\sum_{n=2}^{\infty} a_n \,\Delta z[(z + \Delta z)^{n-2} + 2z(z + \Delta z)^{n-3} + \cdots + (n-2)z^{n-3}(z + \Delta z) + (n-1)z^{n-2}]. \tag{6}$$

The somewhat technical proof of this is given in App. 4.

(c) We consider (6). The brackets contain $n - 1$ terms, and the largest coefficient is $n - 1$. Since $(n-1)^2 \leqq n(n-1)$, we see that for $|z| \leqq R_0$ and $|z + \Delta z| \leqq R_0$, $R_0 < R$, the absolute value of this series (6) cannot exceed

$$|\Delta z| \sum_{n=2}^{\infty} |a_n| n(n-1) R_0^{n-2}. \tag{7}$$

This series with a_n instead of $|a_n|$ is the second derived series of (2) at $z = R_0$ and converges absolutely by Theorem 3 of this section and Theorem 1 of Sec. 15.2. Hence our present series (7) converges. Let the sum of (7) (without the factor $|\Delta z|$) be $K(R_0)$. Since (6) is the right side of (5), our present result is

$$\left| \frac{f(z + \Delta z) - f(z)}{\Delta z} - f_1(z) \right| \leqq |\Delta z| K(R_0).$$

Letting $\Delta z \to 0$ and noting that $R_0 \, (< R)$ is arbitrary, we conclude that $f(z)$ is analytic at any point interior to the circle of convergence and its derivative is represented by the derived series. From this the statements about the higher derivatives follow by induction. ■

Summary. The results in this section show that power series are about as nice as we could hope for: we can differentiate and integrate them term by term (Theorems 3 and 4). Theorem 5 accounts for the great importance of power series in complex analysis: the sum of such a series (with a positive radius of convergence) is an analytic function and has derivatives of all orders, which thus in turn are analytic functions. But this is only part of the story. In the next section we show that, conversely, *every* given analytic function $f(z)$ can be represented by power series, called ***Taylor series*** and being the complex analog of the real Taylor series of calculus.

PROBLEM SET 15.3

1. **Relation to Calculus.** Material in this section generalizes calculus. Give details.

2. **Termwise addition.** Write out the details of the proof on termwise addition and subtraction of power series.

3. **On Theorem 3.** Prove that $\sqrt[n]{n} \to 1$ as $n \to \infty$, as claimed.

4. **Cauchy product.** Show that $(1 - z)^{-2} = \sum_{n=0}^{\infty} (n + 1)z^n$ (a) by using the Cauchy product, (b) by differentiating a suitable series.

5–15 RADIUS OF CONVERGENCE BY DIFFERENTIATION OR INTEGRATION

Find the radius of convergence in two ways: (a) directly by the Cauchy–Hadamard formula in Sec. 15.2, and (b) from a series of simpler terms by using Theorem 3 or Theorem 4.

5. $\displaystyle\sum_{n=2}^{\infty} \frac{n(n-1)}{4^n}(z - 2i)^n$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{z}{2\pi}\right)^{2n+1}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{5^n}(z + 2i)^{2n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n(n+1)} z^n$

9. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n(n+1)(n+2)} z^{2n}$

10. $\sum_{n=k}^{\infty} \binom{n}{k} \left(\frac{z}{2}\right)^n$

11. $\sum_{n=1}^{\infty} \frac{2^n n(n+1)}{5^n} z^{2n}$

12. $\sum_{n=1}^{\infty} \frac{2n(2n-1)}{n^n} z^{2n-2}$

13. $\sum_{n=0}^{\infty} \left[\binom{n+k}{k}\right]^{-1} z^{n+k}$

14. $\sum_{n=0}^{\infty} \binom{n+m}{m} z^n$

15. $\sum_{n=2}^{\infty} \frac{5^n n(n-1)}{3^n} (z-i)^n$

16–20 APPLICATIONS OF THE IDENTITY THEOREM

State clearly and explicitly where and how you are using Theorem 2.

16. **Even functions.** If $f(z)$ in (2) is *even* (i.e., $f(-z) = f(z)$), show that $a_n = 0$ for odd n. Give examples.

17. **Odd function.** If $f(z)$ in (2) is *odd* (i.e., $f(-z) = -f(z)$), show that $a_n = 0$ for even n. Give examples.

18. **Binomial coefficients.** Using $(1+z)^p(1+z)^q = (1+z)^{p+q}$, obtain the basic relation

$$\sum_{n=0}^{r} \binom{p}{n}\binom{q}{r-n} = \binom{p+q}{r}.$$

19. Find applications of Theorem 2 in differential equations and elsewhere.

20. **TEAM PROJECT. Fibonacci numbers.**[2] **(a)** The Fibonacci numbers are recursively defined by $a_0 = a_1 = 1$, $a_{n+1} = a_n + a_{n-1}$ if $n = 1, 2, \cdots$. Find the limit of the sequence (a_{n+1}/a_n).

(b) Fibonacci's rabbit problem. Compute a list of $a_1, \cdots, a_{12}$. Show that $a_{12} = 233$ is the number of pairs of rabbits after 12 months if initially there is 1 pair and each pair generates 1 pair per month, beginning in the second month of existence (no deaths occurring).

(c) Generating function. Show that the *generating function* of the **Fibonacci numbers** is $f(z) = 1/(1 - z - z^2)$; that is, if a power series (1) represents this $f(z)$, its coefficients must be the Fibonacci numbers and conversely. *Hint.* Start from $f(z)(1 - z - z^2) = 1$ and use Theorem 2.

15.4 Taylor and Maclaurin Series

The **Taylor series**[3] of a function $f(z)$, the complex analog of the real Taylor series is

(1) $$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n \quad \text{where} \quad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

or, by (1), Sec. 14.4,

(2) $$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*.$$

In (2) we integrate counterclockwise around a simple closed path C that contains z_0 in its interior and is such that $f(z)$ is analytic in a domain containing C and every point inside C.

A **Maclaurin series**[3] is a Taylor series with center $z_0 = 0$.

[2] LEONARDO OF PISA, called FIBONACCI (= son of Bonaccio), about 1180–1250, Italian mathematician, credited with the first renaissance of mathematics on Christian soil.

[3] BROOK TAYLOR (1685–1731), English mathematician who introduced real Taylor series. COLIN MACLAURIN (1698–1746), Scots mathematician, professor at Edinburgh.

PROBLEM SET 15.4

1. **Calculus.** Which of the series in this section have you discussed in calculus? What is new?
2. **On Examples 5 and 6.** Give all the details in the derivation of the series in those examples.

3–10 MACLAURIN SERIES

Find the Maclaurin series and its radius of convergence.

3. $\sin \frac{z^2}{2}$
4. $\frac{z+2}{1-z^2}$
5. $\frac{1}{8+z^4}$
6. $\frac{1}{1+2iz}$
7. $2\sin^2(z/2)$
8. $\sin^2 z$
9. $\int_0^z \exp(-t^2)\,dt$
10. $\exp(z^2)\int_0^z \exp(-t^2)\,dt$

11–14 HIGHER TRANSCENDENTAL FUNCTIONS

Find the Maclaurin series by termwise integrating the integrand. (The integrals cannot be evaluated by the usual methods of calculus. They define the **error function** erf z, **sine integral** Si(z), and **Fresnel integrals**[4] S(z) and C(z), which occur in statistics, heat conduction, optics, and other applications. These are special so-called higher transcendental functions.)

11. $S(z) = \int_0^z \sin t^2\,dt$
12. $C(z) = \int_0^z \cos t^2\,dt$
13. $\operatorname{erf} z = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt$
14. $\operatorname{Si}(z) = \int_0^z \frac{\sin t}{t}\,dt$

15. **CAS Project. sec, tan. (a) Euler numbers.** The Maclaurin series

$$\sec z = E_0 - \frac{E_2}{2!}z^2 + \frac{E_4}{4!}z^4 - + \cdots \tag{21}$$

defines the *Euler numbers* E_{2n}. Show that $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$. Write a program that computes the E_{2n} from the coefficient formula in (1) or extracts them as a list from the series. (For tables see Ref. [GenRef1], p. 810, listed in App. 1.)

(b) Bernoulli numbers. The Maclaurin series

$$\frac{z}{e^z - 1} = 1 + B_1 z + \frac{B_2}{2!}z^2 + \frac{B_3}{3!}z^3 + \cdots \tag{22}$$

defines the *Bernoulli numbers* B_n. Using undetermined coefficients, show that

$$B_1 = -\tfrac{1}{2},\quad B_2 = \tfrac{1}{6},\quad B_3 = 0,$$
$$B_4 = -\tfrac{1}{30},\quad B_5 = 0,\quad B_6 = \tfrac{1}{42},\cdots. \tag{23}$$

Write a program for computing B_n.

(c) Tangent. Using (1), (2), Sec. 13.6, and (22), show that $\tan z$ has the following Maclaurin series and calculate from it a table of $B_0, \cdots, B_{20}$:

$$\tan z = \frac{2i}{e^{2iz}-1} - \frac{4i}{e^{4iz}-1} - i = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{2^{2n}(2^{2n}-1)}{(2n)!}B_{2n}\,z^{2n-1}. \tag{24}$$

16. **Inverse sine.** Developing $1/\sqrt{1-z^2}$ and integrating, show that

$$\arcsin z = z + \left(\frac{1}{2}\right)\frac{z^3}{3} + \left(\frac{1\cdot 3}{2\cdot 4}\right)\frac{z^5}{5} + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\frac{z^7}{7} + \cdots \quad (|z| < 1).$$

Show that this series represents the principal value of $\arcsin z$ (defined in Team Project 30, Sec. 13.7).

17. **TEAM PROJECT. Properties from Maclaurin Series.** Clearly, from series we can compute function values. In this project we show that properties of functions can often be discovered from their Taylor or Maclaurin series. Using suitable series, prove the following.

(a) The formulas for the derivatives of e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$, and $\operatorname{Ln}(1+z)$

(b) $\frac{1}{2}(e^{iz} + e^{-iz}) = \cos z$

(c) $\sin z \neq 0$ for all pure imaginary $z = iy \neq 0$

18–25 TAYLOR SERIES

Find the Taylor series with center z_0 and its radius of convergence.

18. $1/z, \quad z_0 = i$
19. $1/(1+z), \quad z_0 = -i$
20. $\cos^2 z, \quad z_0 = \pi/2$
21. $\cos z, \quad z_0 = \pi$
22. $\cosh(z - \pi i), \quad z_0 = \pi i$
23. $1/(z-i)^2, \quad z_0 = -i$
24. $e^{z(z-2)}, \quad z_0 = 1$
25. $\sinh(2z - i), \quad z_0 = i/2$

[4] AUGUSTIN FRESNEL (1788–1827), French physicist and engineer, known for his work in optics.

CHAPTER 15 REVIEW QUESTIONS AND PROBLEMS

1. What is convergence test for series? State two tests from memory. Give examples.
2. What is a power series? Why are these series very important in complex analysis?
3. What is absolute convergence? Conditional convergence? Uniform convergence?
4. What do you know about convergence of power series?
5. What is a Taylor series? Give some basic examples.
6. What do you know about adding and multiplying power series?
7. Does every function have a Taylor series development? Explain.
8. Can properties of functions be discovered from Maclaurin series? Give examples.
9. What do you know about termwise integration of series?
10. How did we obtain Taylor's formula from Cauchy's formula?

11–15 RADIUS OF CONVERGENCE

Find the radius of convergence.

11. $\sum_{n=0}^{\infty} (z+1)^n$
12. $\sum_{n=2}^{\infty} \frac{4^n}{n-1}(z-\pi i)^n$
13. $\sum_{n=2}^{\infty} \frac{n(n-1)}{4^n}(z-i)^n$
14. $\sum_{n=1}^{\infty} \frac{n^5}{n!}(z-3i)^{2n}$
15. $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{3n} z^n$

16–20 RADIUS OF CONVERGENCE

Find the radius of convergence. Try to identify the sum of the series as a familiar function.

16. $\sum_{n=1}^{\infty} \frac{z^n}{n}$
17. $\sum_{n=0}^{\infty} \frac{(-2)^n}{n!} z^n$
18. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\pi z)^{2n+1}$
19. $\sum_{n=0}^{\infty} \frac{z^{n+1/2}}{(2n+1)!}$
20. $\sum_{n=0}^{\infty} \frac{z^n}{(3+4i)^n}$

21–25 MACLAURIN SERIES

Find the Maclaurin series and its radius of convergence. Show details.

21. $\cosh z^2$
22. $1/(1-z)^3$
23. $\cos(z^2)$
24. $1/(\pi z+1)$
25. $(e^{z^2}-1)/z^2$

26–30 TAYLOR SERIES

Find the Taylor series with the given point as center and its radius of convergence.

26. z^5, i
27. $\sin z$, π
28. $1/z$, $2i$
29. $\mathrm{Ln}\, z$, 3
30. e^z, πi

SUMMARY OF CHAPTER 15
Power Series, Taylor Series

Sequences, series, and convergence tests are discussed in Sec. 15.1. A **power series** is of the form (Sec. 15.2)

$$(1) \quad \sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots;$$

z_0 is its *center*. The series (1) converges for $|z-z_0| < R$ and diverges for $|z-z_0| > R$, where R is the **radius of convergence**. Some power series converge