where the outer integral (over C_1) is taken counterclockwise and the inner clockwise, as indicated in Fig. 359.

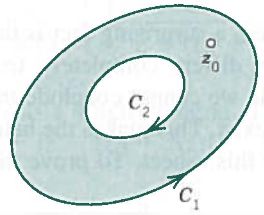


Fig. 359. Formula (3)

PROBLEM SET 14.3

1–4 CONTOUR INTEGRATION

Integrate $z^2/(z^2-1)$ by Cauchy's formula counterclockwise around the circle.

1. $|z+1| = 3/2$

2. $|z-1-i| = \pi/2$

3. $|z+i| = 1.41$

4. $|z+5-5i| = 7$

5–8 Integrate the given function around the unit circle.

5. $(\cos 2z)/4z$

6. $e^{2z}/(\pi z - i)$

7. $z^2/(4z-i)$

8. $(z \sin z)/(2z-1)$

9. CAS EXPERIMENT. Experiment to find out to what extent your CAS can do contour integration. For this, use **(a)** the second method in Sec. 14.1 and **(b)** Cauchy's integral formula.

10. TEAM PROJECT. Cauchy's Integral Theorem. Gain additional insight into the proof of Cauchy's integral theorem by producing (2) with a contour enclosing z_0 (as in Fig. 356) and taking the limit as in the text. Choose

(a) $\displaystyle\oint_C \frac{z^3-6}{z-\frac{1}{2}i}\,dz,$ **(b)** $\displaystyle\oint_C \frac{\sin z}{z-\frac{1}{2}\pi}\,dz,$

and **(c)** another example of your choice.

11–19 FURTHER CONTOUR INTEGRALS

Integrate counterclockwise or as indicated. Show the details.

11. $\displaystyle\oint_C \frac{dz}{z^2+4}$, C: $4x^2 + (y-2)^2 = 4$

12. $\displaystyle\oint_C \frac{z}{z^2+4z+3}\,dz$, C the circle with center -1 and radius 2

13. $\displaystyle\oint_C \frac{z+2}{z-2}\,dz$, C: $|z-1| = 2$

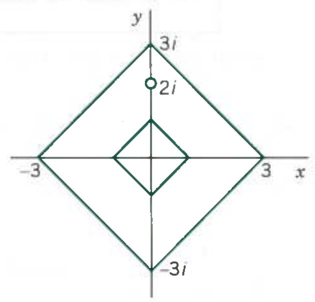
14. $\displaystyle\oint_C \frac{e^z}{ze^z - 2iz}\,dz$, C: $|z| = 0.6$

15. $\displaystyle\oint_C \frac{\cosh (z^2 - \pi i)}{z - \pi i}\,dz$, C the boundary of the square with vertices ± 2, ± 2, $\pm 4i$.

16. $\displaystyle\oint_C \frac{\tan z}{z-i}\,dz$, C the boundary of the triangle with vertices 0 and $\pm 1 + 2i$.

17. $\displaystyle\oint_C \frac{\operatorname{Ln}(z+1)}{z^2+1}\,dz$, C: $|z-i| = 1.4$

18. $\displaystyle\oint_C \frac{\sin z}{4z^2 - 8iz}\,dz$, C consists of the boundaries of the squares with vertices ± 3, $\pm 3i$ counterclockwise and ± 1, $\pm i$ clockwise (see figure).



Problem 18

19. $\displaystyle\oint_C \frac{\exp z^2}{z^2(z-1-i)}\,dz$, C consists of $|z| = 2$ counterclockwise and $|z| = 1$ clockwise.

20. Show that $\displaystyle\oint_C (z-z_1)^{-1}(z-z_2)^{-1}\,dz = 0$ for a simple closed path C enclosing z_1 and z_2, which are arbitrary.

PROOF By assumption, $|f(z)|$ is bounded, say, $|f(z)| < K$ for all z. Using (2), we see that $|f'(z_0)| < K/r$. Since $f(z)$ is entire, this holds for every r, so that we can take r as large as we please and conclude that $f'(z_0) = 0$. Since z_0 is arbitrary, $f'(z) = u_x + iv_x = 0$ for all z (see (4) in Sec. 13.4), hence $u_x = v_x = 0$, and $u_y = v_y = 0$ by the Cauchy–Riemann equations. Thus $u = \text{const}$, $v = \text{const}$, and $f = u + iv = \text{const}$ for all z. This completes the proof. ∎

Another very interesting consequence of Theorem 1 is

THEOREM 3 **Morera's[2] Theorem (Converse of Cauchy's Integral Theorem)**

If $f(z)$ is continuous in a simply connected domain D and if

$$\oint_C f(z)\,dz = 0 \tag{3}$$

for every closed path in D, then $f(z)$ is analytic in D.

PROOF In Sec. 14.2 we showed that if $f(z)$ is analytic in a simply connected domain D, then

$$F(z) = \int_{z_0}^{z} f(z^*)\,dz^*$$

is analytic in D and $F'(z) = f(z)$. In the proof we used only the continuity of $f(z)$ and the property that its integral around every closed path in D is zero; from these assumptions we concluded that $F(z)$ is analytic. By Theorem 1, the derivative of $F(z)$ is analytic, that is, $f(z)$ is analytic in D, and Morera's theorem is proved. ∎

This completes Chapter 14.

PROBLEM SET 14.4

1–7 CONTOUR INTEGRATION. UNIT CIRCLE

Integrate counterclockwise around the unit circle.

1. $\displaystyle\oint_C \frac{\sin 2z}{z^4}\,dz$

2. $\displaystyle\oint_C \frac{z^6}{(2z-1)^6}\,dz$

3. $\displaystyle\oint_C \frac{e^{-z}}{z^n}\,dz, \quad n = 1, 2, \cdots$

4. $\displaystyle\oint_C \frac{e^z \cos z}{(z-\pi/4)^3}\,dz$

5. $\displaystyle\oint_C \frac{\sinh 2z}{(z-\frac{1}{2})^4}\,dz$

6. $\displaystyle\oint_C \frac{dz}{(z-2i)^2(z-i/2)^2}$

7. $\displaystyle\oint_C \frac{\cos z}{z^{2n+1}}\,dz, \quad n = 0, 1, \cdots$

8–19 INTEGRATION. DIFFERENT CONTOURS

Integrate. Show the details. *Hint.* Begin by sketching the contour. Why?

8. $\displaystyle\oint_C \frac{z^3 + \sin z}{(z-i)^3}\,dz$, C the boundary of the square with vertices ± 2, $\pm 2i$ counterclockwise.

9. $\displaystyle\oint_C \frac{\tan \pi z}{z^2}\,dz$, C the ellipse $16x^2 + y^2 = 1$ clockwise.

10. $\displaystyle\oint_C \frac{4z^3 - 6}{z(z-1-i)^2}\,dz$, C consists of $|z| = 3$ counterclockwise and $|z| = 1$ clockwise.

---

[2]GIACINTO MORERA (1856–1909), Italian mathematician who worked in Genoa and Turin.

11. $\oint_C \frac{(1+z)\cos z}{(2z-1)^2}\,dz$, C: $|z - i| = 2$ counterclockwise.

12. $\oint_C \frac{\exp(z^2)}{z(z-2i)^2}\,dz$, C: $z - 3i| = 2$ clockwise.

13. $\oint_C \frac{\operatorname{Ln} z}{(z-4)^2}\,dz$, C: $|z - 3| = 2$ counterclockwise.

14. $\oint_C \frac{\operatorname{Ln}(z+3)}{(z-2)(z+1)^2}\,dz$, C the boundary of the square with vertices ± 1.5, $\pm 1.5i$, counterclockwise.

15. $\oint_C \frac{\cosh 4z}{(z-4)^3}\,dz$, C consists of $|z| = 6$ counterclockwise and $|z - 3| = 2$ clockwise.

16. $\oint_C \frac{e^{4z}}{z(z-2i)^2}\,dz$, C consists of $|z - i| = 3$ counterclockwise and $|z| = 1$ clockwise.

17. $\oint_C \frac{e^{-z}\sin z}{(z-4)^3}\,dz$, C consists of $|z| = 5$ counterclockwise and $|z - 3| = \frac{3}{2}$ clockwise.

18. $\oint_C \frac{\sinh z}{z^n}\,dz$, C: $|z| = 1$ counterclockwise, n integer.

19. $\oint_C \frac{e^{3z}}{(4z - \pi i)^3}\,dz$, C: $|z| = 1$, counterclockwise.

20. **TEAM PROJECT. Theory on Growth**

(a) Growth of entire functions. If $f(z)$ is not a constant and is analytic for all (finite) z, and R and M are any positive real numbers (no matter how large), show that there exist values of z for which $|z| > R$ and $|f(z)| > M$. *Hint.* Use Liouville's theorem.

(b) Growth of polynomials. If $f(z)$ is a polynomial of degree $n > 0$ and M is an arbitrary positive real number (no matter how large), show that there exists a positive real number R such that $|f(z)| > M$ for all $|z| > R$.

(c) Exponential function. Show that $f(z) = e^x$ has the property characterized in (a) but does not have that characterized in (b).

(d) Fundamental theorem of algebra. *If $f(z)$ is a polynomial in z, not a constant, then $f(z) = 0$ for at least one value of z.* Prove this. *Hint.* Use (a).

CHAPTER 14 REVIEW QUESTIONS AND PROBLEMS

1. What is a parametric representation of a curve? What is its advantage?
2. What did we assume about paths of integration $z = z(t)$? What is $\dot{z} = dz/dt$ geometrically?
3. State the definition of a complex line integral from memory.
4. Can you remember the relationship between complex and real line integrals discussed in this chapter?
5. How can you evaluate a line integral of an analytic function? Of an arbitrary continous complex function?
6. What value do you get by counterclockwise integration of $1/z$ around the unit circle? You should remember this. It is basic.
7. Which theorem in this chapter do you regard as most important? State it precisely from memory.
8. What is independence of path? Its importance? State a basic theorem on independence of path in complex.
9. What is deformation of path? Give a typical example.
10. Don't confuse Cauchy's integral theorem (also known as ***Cauchy–Goursat theorem***) and Cauchy's integral formula. State both. How are they related?
11. What is a doubly connected domain? How can you extend Cauchy's integral theorem to it?
12. What do you know about derivatives of analytic functions?
13. How did we use integral formulas for derivatives in evaluating integrals?
14. How does the situation for analytic functions differ with respect to derivatives from that in calculus?
15. What is Liouville's theorem? To what complex functions does it apply?
16. What is Morera's theorem?
17. If the integrals of a function $f(z)$ over each of the two boundary circles of an annulus D taken in the same sense have different values, can $f(z)$ be analytic everywhere in D? Give reason.
18. Is $\operatorname{Im} \oint_C f(z)\,dz = \oint_C \operatorname{Im} f(z)\,dz$? Give reason.
19. Is $\left|\oint_C f(z)\,dz\right| = \oint_C |f(z)|\,dz$?
20. How would you find a bound for the left side in Prob. 19?

21–30 INTEGRATION

Integrate by a suitable method.

21. $\int_C z\cosh(z^2)\,dz$ from 0 to $\pi i/2$.

If the sequence of the roots in (9) and (10) converges, we more conveniently have

THEOREM 10

Root Test

If a series $z_1 + z_2 + \cdots$ is such that $\lim_{n\to\infty} \sqrt[n]{|z_n|} = L$, then:

(a) *The series converges absolutely if $L < 1$.*

(b) *The series diverges if $L > 1$.*

(c) *If $L = 1$, the test fails; that is, no conclusion is possible.*

PROBLEM SET 15.1

1–10 SEQUENCES

Is the given sequence $z_1, z_2, \cdots, z_n, \cdots$ bounded? Convergent? Find its limit points. Show your work in detail.

1. $z_n = (1 + i)^{2n}/2^n$

2. $z_n = (1 + 2i)^n/n!$

3. $z_n = n\pi/(2 + 4ni)$

4. $z_n = (2 - i)^n$

5. $z_n = (-1)^n + 5i$

6. $z_n = (\cos 2n\pi i)/n$

7. $z_n = n^2 - i/2n^2$

8. $z_n = [(1 + 2i)/\sqrt{5}]^n$

9. $z_n = (2 + 2i)^{-n}$

10. $z_n = \sin(\frac{1}{4}n\pi) + i^n$

11. CAS EXPERIMENT. Sequences. Write a program for graphing complex sequences. Use the program to discover sequences that have interesting "geometric" properties, e.g., lying on an ellipse, spiraling to its limit, having infinitely many limit points, etc.

12. Addition of sequences. If $z_1, z_2, \cdots$ converges with the limit l and $z_1^*, z_2^*, \cdots$ converges with the limit l^*, show that $z_1 + z_1^*, z_2 + z_2^*, \cdots$ is convergent with the limit $l + l^*$.

13. Bounded sequence. Show that a complex sequence is bounded if and only if the two corresponding sequences of the real parts and of the imaginary parts are bounded.

14. On Theorem 1. Illustrate Theorem 1 by an example of your own.

15. On Theorem 2. Give another example illustrating Theorem 2.

16–25 SERIES

Is the given series convergent or divergent? Give a reason. Show details.

16. $\displaystyle\sum_{n=0}^{\infty} \frac{(20 + 30i)^n}{n!}$

17. $\displaystyle\sum_{n=2}^{\infty} \frac{(-i)^n}{\ln n}$

18. $\displaystyle\sum_{n=1}^{\infty} n^2\left(\frac{i}{4}\right)^n$

19. $\displaystyle\sum_{n=0}^{\infty} \frac{i^n}{n^2 - i}$

20. $\displaystyle\sum_{n=0}^{\infty} \frac{n + i}{3n^2 + 2i}$

21. $\displaystyle\sum_{n=0}^{\infty} \frac{(\pi + \pi i)^{2n+1}}{(2n + 1)!}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n}}$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(1 + i)^{2n}}{(2n)!}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{(3i)^n n!}{n^n}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

26. Significance of (7). What is the difference between (7) and just stating $|z_{n+1}/z_n| < 1$?

27. On Theorems 7 and 8. Give another example showing that Theorem 7 is more general than Theorem 8.

28. CAS EXPERIMENT. Series. Write a program for computing and graphing numeric values of the first n partial sums of a series of complex numbers. Use the program to experiment with the rapidity of convergence of series of your choice.

29. Absolute convergence. Show that if a series converges absolutely, it is convergent.

30. Estimate of remainder. Let $|z_{n+1}/z_n| \leqq q < 1$, so that the series $z_1 + z_2 + \cdots$ converges by the ratio test. Show that the remainder $R_n = z_{n+1} + z_{n+2} + \cdots$ satisfies the inequality $|R_n| \leqq |z_{n+1}|/(1 - q)$. Using this, find how many terms suffice for computing the sum s of the series

$$\sum_{n=1}^{\infty} \frac{n + i}{2^n n}$$

with an error not exceeding 0.05 and compute s to this accuracy.

EXAMPLE 6 **Extension of Theorem 2**

Find the radius of convergence R of the power series

$$\sum_{n=0}^{\infty}\left[1+(-1)^n+\frac{1}{2^n}\right]z^n = 3+\frac{1}{2}z+\left(2+\frac{1}{4}\right)z^2+\frac{1}{8}z^3+\left(2+\frac{1}{16}\right)z^4+\cdots.$$

Solution. The sequence of the ratios $\frac{1}{6}$, $2(2+\frac{1}{4})$, $1/(8(2+\frac{1}{4}))$, $\cdots$ does not converge, so that Theorem 2 is of no help. It can be shown that

$$(6^*) \qquad \tilde{R} = 1/\tilde{L}, \qquad \tilde{L} = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

This still does not help here, since $(\sqrt[n]{|a_n|})$ does not converge because $\sqrt[n]{|a_n|} = \sqrt[n]{1/2^n} = \frac{1}{2}$ for odd n, whereas for even n we have

$$\sqrt[n]{|a_n|} = \sqrt[n]{2+1/2^n} \to 1 \quad \text{as} \quad n\to\infty,$$

so that $\sqrt[n]{|a_n|}$ has the two limit points $\frac{1}{2}$ and 1. It can further be shown that

$$(6^{**}) \qquad R = 1/\tilde{l}, \qquad \tilde{l} \text{ the greatest limit point of the sequence } \left\{\sqrt[n]{|a_n|}\right\}.$$

Here $\tilde{l} = 1$, so that $R = 1$. *Answer.* The series converges for $|z| < 1$. ■

Summary. Power series converge in an open circular disk or some even for every z (or some only at the center, but they are useless); for the radius of convergence, see (6) or Example 6.

Except for the useless ones, power series have sums that are analytic functions (as we show in the next section); this accounts for their importance in complex analysis.

PROBLEM SET 15.2

1. **Power series.** Are $1/z + z + z^2 + \cdots$ and $z + z^{3/2} + z^2 + z^3 + \cdots$ power series? Explain.
2. **Radius of convergence.** What is it? Its role? What motivates its name? How can you find it?
3. **Convergence.** What are the only basically different possibilities for the convergence of a power series?
4. **On Examples 1–3.** Extend them to power series in powers of $z - 4 + 3\pi i$. Extend Example 1 to the case of radius of convergence 6.
5. **Powers z^{2n}.** Show that if $\Sigma a_n z^n$ has radius of convergence R (assumed finite), then $\Sigma a_n z^{2n}$ has radius of convergence $\sqrt{R}$.

6–18 RADIUS OF CONVERGENCE

Find the center and the radius of convergence.

6. $\sum_{n=0}^{\infty} 2^n(z-1)^n$
7. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(z-\frac{1}{4}\pi\right)^{2n}$
8. $\sum_{n=0}^{\infty} \frac{n^n}{n!}(z-\pi i)^n$
9. $\sum_{n=0}^{\infty} \frac{n(n-1)}{2^n}(z+i)^{2n}$
10. $\sum_{n=0}^{\infty} \frac{(z-2i)^n}{n^n}$
11. $\sum_{n=0}^{\infty} \left(\frac{3-i}{5+2i}\right)z^n$
12. $\sum_{n=0}^{\infty} \frac{(-1)^n n}{8^n} z^n$
13. $\sum_{n=0}^{\infty} 16^n(z+i)^{4n}$
14. $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{2n}(n!)^2} z^{2n}$
15. $\sum_{n=0}^{\infty} \frac{(2n)!}{4^n(n!)^2}(z-2i)^n$
16. $\sum_{n=0}^{\infty} \frac{(3n)!}{2^n(n!)^3} z^n$
17. $\sum_{n=1}^{\infty} \frac{3^n}{n(n+1)} z^{2n+1}$
18. $\sum_{n=0}^{\infty} \frac{2(-1)^n}{\sqrt{\pi}(2n+1)n!} z^{2n+1}$
19. **CAS PROJECT. Radius of Convergence.** Write a program for computing R from (6), (6*), or (6**), in