TMA4220 Calculus 4K, Midterm Practice problems, Fall 2016

Laplace transform, Fourier series and Fourier transform, PDEs.

Problem 1. Sketch the graph of the function $g(t) = (t-3)u_2(t)-(t-2)u_3(t)$ and find the Laplace transform of g.

Problem 2. Solve the initial value problem $y'' - 2y' + 2y = \delta(t-1) + e^{-t}$, $y(0) = 1,\ y'(0) = 0$, using the Laplace transform.

Problem 3. Consider the linear system of differential equations $x' = x + y, \quad y' = 4x + y$ with initial conditions $x(0) = 0$ and $y(0) = 2$ and let $X(s) = \mathcal{L}\{x\}(s)$ and $Y(s) = \mathcal{L}\{y\}(s)$ be the Laplace transforms of the functions $x(t)$ and $y(t)$, respectively.
a) Find $X(s)$ and $Y(s)$. b) Determine $x(t)$ and $y(t)$.

Problem 4. Use the Laplace transform to solve the equation $y'(t)+\int_0^t e^u y(t-u)du = t+y(t),\ t > 0$ with initial condition $y(0) = 1$.

Problem 5. Let $f(x) = x^2$ when $0 < x < 1$.
a) Find the coefficients of the sine Fourier series for f, $S_f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$. Sketch the graph of S_f on $[0, 3]$ and determine the values $S_f(0), S_f(1/2), S_f(1)$.
b) Find the coefficients of the corresponding cosine Fourier series of f, $C_f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$.and sketch the graph of the C_f on $[0, 3]$.
c) If you want approximate the function f by a partial sum F of one of the series from a) and b) such that the square error $\int_0^1 (f-F)^2 dx$ is less than 10^{-5}, which series would you use and why? How many terms would you need (a rough approximation)? Why do the coefficients of sine and cosine series behave so differently?

Problem 6. Let $f(x) = x^2(\pi - x)$ when $0 < x < \pi$, we extend f to a π-periodic function.
a) Find the complex Fourier series of f and convert this series to real form.
b) Find a particular solution of the differential equation $y'' + 3y' + 2y = f(x)$.

Problem 7. a) Let $\hat{f}(w)$ be the Fourier transform of some function $f(x)$. Compute the Fourier transform of $f(x-a)$ (show your work).
b) Find the Fourier transform of e^{-x^2+2px} (use your computation from a) and the formula $\mathcal{F}(e^{-kx^2}) = (2k)^{-1/2}e^{-w^2/(4k)}$).

Problem 8. Let $f(x) = 1$ when $-1 < x < 1$ and $f(x) = 0$ when $|x| > 1$.
a) Compute $\hat{f}(w)$.
b) Compute $h(x) = (f * f)(x) = \int_{-\infty}^{\infty} f(y)f(x-y)dy$.
c) Find $\hat{h}$ by the definition and then by the convolution theorem.

Problem 9. Consider the wave equation $u_{tt} = c^2 u_{xx}$ for $0 < x < \pi,\ t > 0$ with boundary condition $u(0, t) = u(\pi, t) = 0$ and initial data $u(x, 0) = 0$, $u_t(x, 0) = \sin x(1 + \cos x)$.
a) Find all functions of the form $u(x, t) = F(x)G(t)$ that solve the equation and satisfy the boundary conditions.
b) Solve the equation with boundary and initial conditions.

Problem 10. Solve the heat equation on the infinite string, $u_t = c^2 u_{xx}$ for $-\infty < x < \infty$ and $t > 0$ if the initial temperature is given by $u(x, 0) = xe^{-x^2}$. (Hint: you may use that $\mathcal{F}(xe^{-ax^2}) = -\frac{iw}{2a\sqrt{2a}}e^{-w^2/(4a)}$.)