TMA4120, Matematikk 4K, Fall 2016

LECTURE SCHEDULE AND ASSIGNMENTS

Date	**Section**	Topic	HW	Textbook problems	Suppl.	**Answers**
Aug 22 Aug 24/25	6.1 6.2-3	Laplace transform ODE, Heaviside function	**1**	**6.1:**1,7,12,21,22,23,25,26,41 **6.2:**9,27; **6.3**:5,13,40	A	**Sept 5**
Aug 29 Aug 31/ 1	6.4-5 6.6-7	Dirac delta Convolution Systems of ODE	**2**	**6.4:**5,14abc;**6.5:**7,13,19,16e **6.6:**16,17;**6.7:**13; **6.R:**39	B C	**Sept 12**
Sept 5 Sept 7/8	11.1 11.2, 13.1-2	Fourier Series Complex numbers	**3**	**11.1:**9,14,19,21;**11.2:**11,17,25 **13.1:**9,12,18,19;**13.2:**3,8,21	D	**Sept 19**
Sept 12 Sept 14/15	11.4* 11.4, 11.3	Complex Fourier series Approximation Forced oscillations	**4**	**11.3:**15,19;**11.4*:**9,10,13 **11.4:**4,8,11;**11:R:**15,17	E,F,G	**Sept 26**
Sept 19 Sept 21/22	11.7 11.9	Fourier Integral Fourier Transform	**5**	**11.7:**2,11,19;**11.9:**4,9	H,I,J K,L,M	**Oct 3**
Sept 26 Sept 28/29	12.1-3 12.5-6	PDE, Wave Equation Heat Equation	**6**	**12.1:**3,9,15;**12.3:**1,7,15,16,17 **12.6:**5,21	N O	**Oct 10**
Oct 3 Oct 5/6	12.7, 12.4 13.3	Heat and wave eqn. Analytic functions	**7**	**12.4:**13;**12.7:**2,13 **13.3:**3,7,8,10,11,14,21,23	P Q	**Oct 17**
Oct 10 Oct 12/13	13.4-5 13.6-7	Cauchy-Riemann eqns Exponential Trig fns, Log	**8**	**13.4:**3,9,13;**13.5:**5,16 **13.6:**3,9,18; **13.7:**7,17,19,23	R S	**Oct 24**
Oct 17 Oct 19/20	17.1 14.1-2	Conformal mappings Complex integration Cauchy Theorem	**9**	**17.1:**5,8,11,13,15 **14.1:**4,7,12,17,21,23,35;**14.2:**14,22,27		**Oct 31**
Oct 24 Oct 26/27	14.3-4 15.1-2	Cauchy formula Derivatives Power Series	**10**	**14.3:**3,12,18;**14.4:**3,4,8,15 **15.1:**1,2,16,17,19,30;**15.2:**5,6,9		**Nov 7**
Oct 31 Nov 2/3	15.3 15.4(5)	Functions by Power Series Taylor and Maclaurin	**11**	**15.3:**:4,7,16;**15.4:**5,8,9,19,24 **5.R:**14,18,26,29		**Nov 14**
Nov 7 Nov 9/10	16.1 16.2-3	Singularities and zeros Residue Integration	**12**	**16.1:**3,7,13;**16.2:**3,5,6a,7 **16.3:**1,6,9;**16.4:**3,6	T U	**Nov 21**
Nov 14 Nov 16/17	16.4 16.4	Real integrals Applications	13		V,W,X,Y Z,Æ,Ø,Å	**Nov 24**
Nov 21 Nov 23/24		Repetition	-			
Dec 3		**Final Exam**				

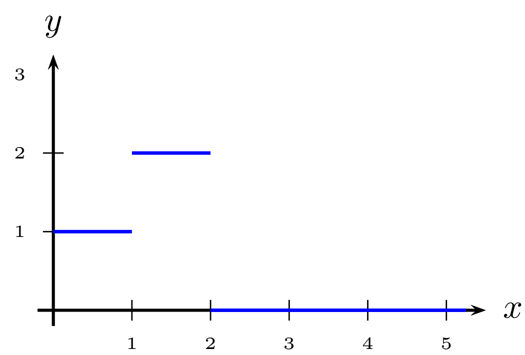
11.4* refers to 11.4 from 9th edition (a scan of the pages can be found on the web-page).
17.1.8: Consider only the lines $x = 2$ and $y = 3$.
17.1.15: Take the cubic polynomial $z^3 + 3z + 4$.
16.2.3: Find all zeros and poles of the given function and determine their orders.

SUPPLEMENTARY PROBLEMS

A. Let $y(t)$ be the solution of the differential equation

$$y''(t) + y'(t) - 2y(t) = r(t)$$

that satisfies $y(0) = 1$, $y'(0) = 1$, where the function $r(t)$ is given by its graph:



Find the Laplace transform of y.

B. Use The Laplace transform to solve the differential equation

$$y'' + 4y' + 4y = 2e^{-2t} + \delta(t-1), \quad t > 0$$

with initial conditions $y(0) = 0$ and $y'(0) = 0$, where δ is the Delta function.

C. Use The Laplace transform to solve the initial value problem

$$y' + y + \int_0^t y(\tau)e^{t-\tau}d\tau = u(t-1), \quad t > 0, \quad y(0) = 1$$

D. The function f is defined by the following conditions:
i) $f(x) = f(-x)$ for all real x.
ii) $f(x) = f(x+4)$ for all real x.
iii) $f(x) = 1 - x$ for $0 < x < 2$.
Sketch the graph of f for $-2 < x < 6$. Find the Fourier series of f.

E. Let $f(x) = x(\pi - x)$ for $0 \le x \le \pi$. The Fourier Sine series of f is given

$$\frac{8}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{(2m+1)^3}.$$

Determine the sum of the series

$$\frac{1}{1^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} - \frac{1}{13^3} - \frac{1}{15^3} + \ldots.$$

F. The 2π-periodic function f is defined by $f(x) = e^x$, $-\pi < x < \pi$.
a) Sketch the graph of the periodic extension f and find the complex Fourier series of f.
b) Determine the sums of the series:

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}, \qquad \sum_{n=2}^{\infty}\frac{1}{1+n^2}$$

G. Let f be the 2π-periodic function defined by $f(x) = x^4$ for $-\pi < x \leq \pi$. The Fourier series of f is

$$\frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{(-1)^n 8(\pi^2 n^2 - 6)}{n^4} \cos(nx).$$

Use this to find the sums of the series:

$$(i) \sum_{n=1}^{\infty} \frac{\pi^2 n^2 - 6}{n^4}, \qquad (ii) \sum_{n=1}^{\infty} \frac{\pi^4 n^4 - 12\pi^2 n^2 + 36}{n^8}.$$

H. Compute the Fourier transform of the function

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ -e^{x}, & x < 0 \end{cases}.$$

Use the result to compute the integral

$$\int_0^{\infty} \frac{w \sin w}{1 + w^2} dw.$$

I. Compute the Fourier transform of the function $f(t) = \cos(t) e^{-t^2}$.

J. Find the Fourier transform of the function $h(x) = e^{-x^2} * e^{-x^2}$. Use the result to express $h(x)$ without an integral or convolution sign.

K. Use the Fourier transform to solve the equation

$$f(x) - \int_{-\infty}^{\infty} e^{-3|x-t|} f(t) dt = e^{-3|x|}.$$

L. Functions $f(x)$ and $g(x)$ are defined by

$$f(x) = \begin{cases} 1 \text{ for } |x| < 1 \\ 0 \text{ otherwise} \end{cases}, \qquad g(x) = \begin{cases} e^{-x} \text{ for } x > 0 \\ 0 \text{ otherwise} \end{cases}.$$

a) Show that the Fourier transforms of f and g are

$$\hat{f}(w) = \sqrt{\frac{2}{\pi}} \frac{\sin w}{w} \text{ and } \hat{g}(w) = \frac{1}{\sqrt{2\pi}} \frac{1 - iw}{1 + w^2}.$$

b) Let $h(x)$ be the convolution of f and g, $h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(x-y) g(y) dy$. Prove that

$$h(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(1 - iw) \sin w}{w(1 + w^2)} e^{iwx} dw$$

and determine the value of the integral

$$\int_{-\infty}^{\infty} \frac{\sin w}{w(1 + w^2)} dw.$$

M. Let $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$. Show that the Fourier transform of f can be written as

$$\hat{f}(w) = \frac{i}{\sqrt{2\pi}} \frac{e^{-iw} - e^{iw}}{w}.$$

Find also the Fourier transform of the convolution $f * f$. Use the inverse Fourier transform to determine the value of the integral

$$\int_{-\infty}^{\infty} \frac{\cos 5w - 2\cos 3w + \cos w}{w^2} dw.$$

(You can use without proof that $f * f$ is a continuous function.)

N. A complex number z_0 satisfies $|e^{z_0}| = 5$. Find the value of $|e^{2z_0+3i}|$.

O. Let a and b be two real constants, consider two boundary value problems

$$(*) \qquad \begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \text{ for } t > 0, \ x \in (0,1) \\ u(0,t) = a \text{ for} t > 0, \\ u(1,t) = b \text{ for} t > 0, \end{cases}$$

and

$$(**) \qquad \begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \text{ for } t > 0, \ x \in (0,1) \\ u(0,t) = 0 \text{ for} t > 0, \\ u(1,t) = 0 \text{ for} t > 0, \end{cases}$$

a) Let u_1 and u_2 be solutions of (*) Determine which boundary value problem are solved by functions $u_1 + u_2$ and $u_1 - u_2$. Does the superposition principle hold for (*) and/or for (**)?
b) Let $u(x,t)$ be a solution of the boundary value problem (*) and let $v(x,t)$ be defined by

$$v(x,t) = u(x,t) - (a + (b-a)x), \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x \in [0,1].$$

Show that $v(x,t)$ is a solution of (**).
Find all solutions of (**) of the form $v(x,t) = F(x)G(t)$.
c) Let $a = -1$ and $b = 1$ in (*). Find a solution of the boundary value problem (*) that satisfies the initial condition $u(x,0) = \sin(\pi x)$ for $0 < x < 1$.

P. a) Find the Fourier Sine Series of the function $f(x) = \pi x - x^2, \ 0 \leq x \leq \pi$.
b) Let $u(x,t)$ be a solution of the boundary value problem

$$\begin{cases} u_t = u_{xx} - 2u_x, & 0 < x < \pi, t > 0, \\ u(0,t) = 0 = u(\pi,t), & t > 0. \end{cases}$$

Show that if $u(x,t) = F(x)G(t)$ then $F(x) = Ce^x \sin nx$ for some integer n.
c) Find a solution $u(x,t)$ of the boundary value problem in b) such that $u(x,0) = e^x f(x), \ 0 < x < \pi$, where $f(x)$ is the function given in a).

Q. Determine which of the following functions are analytic at $z_0 = 1$ $\quad (i)$ $z\text{Re}(z)$, $\quad (ii)$ z^2, $\quad (iii)$ $\frac{1}{z}$.

R. Find all solutions to the equation $e^{2z} = i$.

S. The function $f(z) = y^3 + Bx^2y + iv(x,y)$ is analytic. Determine the constant B and the function $v(x,y)$ if $v(0,0) = 0$. (Hint: Use the Laplaces and Cauchy–Riemann equations.)

T. Find all Laurent series with center $z = 1$ of the function

$$f(z) = \frac{1}{z} + \frac{e^z}{z-1}.$$

U. a) Find all Laurent series with center $z = 0$ of the function

$$f(z) = \frac{1}{z(8z^3 - 1)}$$

and determine the domain of convergence for each series.
b) Let C be the unit circle $|z| = 1$ with positive orientation (counter clockwise). Determine the values of the integrals

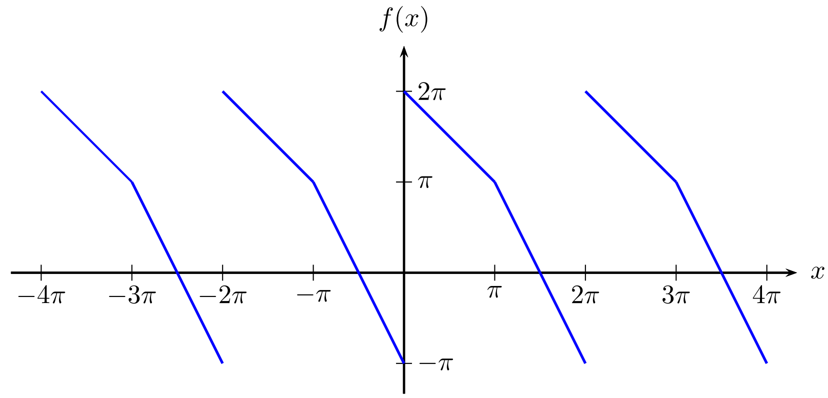
$$\oint_C f(z)dz \qquad \text{and} \qquad \oint_C (\text{Re}z)dz$$

V. Solve the equation

$$y''(t) + y(t) = \begin{cases} 2\sin 2t & \text{for } 0 \le t \le \pi \\ 0 & \text{for } t > \pi, \end{cases}$$

with initial conditions $y(0) = 0,\ y'(0) = 0$.

W. Let $f(x)$ be the 2π-periodic function defined by the graph:



a) Determine the value of the sum of the Fourier series of f at points $x = 0$ and $x = \pi$.
b) Find the Fourier series of f.

X. The temperature in an isolated bar satisfies the equation

$$(*) \qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2u, \quad 0 < x < \pi, \quad t > 0$$

with boundary conditions

$$(**) \qquad \frac{\partial u}{\partial x}(0, t) = 0 = \frac{\partial u}{\partial x}(\pi, t).$$

(The term $2u$ appears because only the ends of the bar are isolated.)
a) Find all solutions of the boundary value problem $(*)$, $(**)$ of the form $u(x,t) = X(x)T(t)$.
b) Use the superposition principle to find the solution satisfying the initial condition

$$u(x, 0) = (\cos(x) + 1)^2, \quad 0 < x < \pi.$$

Y. Determine all values of c such that the function $u(x, y) = e^{cx} \sin y \cos y$ is harmonic.
Find all analytic functions $f(z)$ such that $\text{Re}(f(z)) = u(x, y)$ when $z = x + iy$.

Z. Expand the function $f(z) = \frac{e^{-1/z^2}}{z^2}$ in A Laurent series that converges for $0 < |z| < R$ and find its region of convergence. Compute the residue $\text{Res}_{z=0}\{f(z)\}$.

Æ. It is given that

$$\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2+6x+25}dx = 2\pi i \sum \text{Res}\left\{\frac{e^{2iz}}{z^2+6z+25}\right\},$$

where the sum is taken over the singular points of the function in the upper half-plane. Compute the integral and determine the value of

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2+6x+25}dx.$$

Ø. Use the residue calculus to compute the integral

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}.$$

Å. a) Show that the function $f(z) = \frac{e^{iz}-e^{5iz}}{z^2}$ can be written as

$$f(z) = \frac{b}{z} + g(z) \qquad \text{for} \quad |z| > 0,$$

where b is a complex number and $g(z)$ is an analytic function. Determine the value of b.
b) Let S_R be the half-circle parametrized by $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$. Show that

$$\int_{S_R} f(z)dz \to 4\pi \quad \text{when} \quad R \to 0.$$

c) Show that

$$\int_{S_R} f(z)dz \to 0 \quad \text{when} \quad R \to \infty$$

and compute the integral

$$\int_0^{\infty} \frac{\cos x - \cos 5x}{2x^2}dx.$$

(You may assume that the last integral converges without proving it.)