Løysingsforslag for TMA4120, Øving 5

October 3, 2016

11.7.2) La $f : \mathbb{R} \to \mathbb{R}$ vere gitt av

$$f(x) = \begin{cases} \frac{\pi}{2} \sin x & \text{for } |x| \leq \pi \\ 0 & \text{ellers.} \end{cases}$$

Sidan f er odde, er $A(\omega) = 0$ for alle $\omega \geq 0$. Vi reknar ut

$$\begin{aligned} B(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \omega x dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin x \sin \omega x dx \\ &= \frac{1}{2} \left[\frac{\cos x \sin \omega x - \omega \sin x \cos \omega x}{\omega^2 - 1} \right]_{x=-\pi}^{\pi} \\ &= \frac{\sin \pi\omega}{1 - \omega^2}. \end{aligned}$$

Det er klart at f oppfyller krava for representasjonsteoremet for Fourierintegral side 513. Sjekk at du ser kvifor! Sidan f ikkje har diskontinuitetspunkt, er middelverdien overalt lik funksjonsverdien sjølv. Derfor er

$$\int_0^{\infty} (A(\omega) \cos \omega x + B(\omega) \sin \omega x)\, d\omega = \int_0^{\infty} \frac{\sin \pi\omega}{1 - \omega^2} \sin \omega x d\omega = f(x) = \begin{cases} \frac{\pi}{2} \sin x & \text{for } |x| \leq \pi \\ 0 & \text{ellers.} \end{cases}$$

Det var dette vi skulle vise.

11.7.11) La $g : \mathbb{R} \to \mathbb{R}$ vere gitt ved $g(x) = f(|x|)$. Sidan g tilfredsstiller krava i Teorem 1 og er like, er $g(x)$, og dermed $f(x)$, gitt ved (10) på side 515. Reknar ut $A(w)$:

$$\begin{aligned} A(w) &= \frac{2}{\pi} \int_0^{\infty} g(v) \cos wv dv \\ &= \frac{2}{\pi} \int_0^{\pi} \sin v \cos wv dv \\ &= \frac{2}{\pi} \left[\frac{w \sin v \sin wv + \cos v \cos wv}{w^2 - 1} \right]_0^{\pi} \\ &= \frac{2 \cos w\pi + 2}{\pi(1 - w^2)}. \end{aligned}$$

Vi får dermed

$$f(x) = \int_0^\infty \frac{2\cos w\pi + 2}{\pi(1-w^2)} \cos wx dw$$

11.7.19) Vi bruker same metode som over og latar som om f er ein del av ein odde funksjon definert på heile $\mathbb{R}$. Dermed er f gitt ved (11), der

$$\begin{aligned}
B(w) &= \frac{2}{\pi}\int_0^\infty f(v)\sin wv dv \\
&= \frac{2}{\pi}\int_0^1 e^v \sin wv dv \\
&= \frac{2}{\pi}\left([e^v \sin wv]_0^1 - \int_0^1 we^v \cos wv dv\right) \\
&= \frac{2}{\pi}\left(e\sin w - [e^v w\cos wv]_0^1 - \int_0^1 w^2 e^v \sin wv dv\right) \\
&= \frac{2}{\pi}\left(e\sin w + w - ew\cos w - w^2\int_0^1 e^v \sin wv dv\right) \\
\Rightarrow B(w) &= \frac{2}{\pi}\cdot\frac{w + e(\sin w - w\cos w)}{1+w^2}.
\end{aligned}$$

Det gir

$$f(x) = \int_0^\infty \frac{2(w + e(\sin w - w\cos w))}{\pi(1+w^2)} \sin wx dw$$

11.9.4)

$$\begin{aligned}
\sqrt{2\pi}\hat{f}(w) &= \int_{-\infty}^\infty f(x)e^{-iwx}dx \\
&= \int_{-\infty}^0 e^{kx}e^{-iwx}dx \\
&= \int_{-\infty}^0 e^{(k-iw)x}dx \\
&= \left[\frac{1}{k-iw}e^{(k-iw)x}\right]_{x=-\infty}^0 \\
&= \frac{1}{k-iw} \\
\Rightarrow \hat{f}(w) &= \frac{1}{\sqrt{2\pi}(k-iw)}
\end{aligned}$$

11.9.9)

$$f(x) = \begin{cases} |x| & \text{for } -1 < x < 1 \\ 0 & \text{ellers} \end{cases}$$

$$
\begin{aligned}
\hat{f}(w) &= \frac{1}{\sqrt{2\pi}}\left(\int_{-1}^{0} -xe^{-iwx}dx + \int_{0}^{1} xe^{-iwx}dx\right)\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{iw}xe^{-iwx}\Big|_{-1}^{0} - \int_{-1}^{0}\frac{1}{iw}e^{-iwx}dx - \frac{1}{iw}xe^{-iwx}\Big|_{0}^{1} + \int_{0}^{1}\frac{1}{iw}e^{-iwx}dx\right)\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{iw}e^{iw} + \frac{1}{(iw)^2}e^{-iwx}\Big|_{-1}^{0} - \frac{1}{iw}e^{-iw} - \frac{1}{(iw)^2}e^{-iwx}\Big|_{0}^{1}\right)\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{iw}e^{iw} - \frac{1}{w^2} + \frac{1}{w^2}e^{iw} - \frac{1}{iw}e^{-iw} + \frac{1}{w^2}e^{-iw} - \frac{1}{w^2}\right)\\
&= \frac{1}{\sqrt{2\pi}}\left(-\frac{2}{w^2} + \frac{2}{w^2}w\sin w + \frac{2}{w^2}\cos w\right)\\
&= \frac{\sqrt{2}}{\sqrt{\pi}w^2}\left(\cos w + w\sin w - 1\right)
\end{aligned}
$$

Her brukte vi at $e^{iw} = \cos w + i\sin w$. Det går også an å utnytte at f er like og skrive $\hat{f}(w) = \hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\int_0^1 x\cos wx dx$, der $\hat{f}_c(w)$ er Fourier-cosinus-transformen av f frå 11.8.

H) Vi har

$$
\begin{aligned}
\hat{f}(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx\\
&= \frac{1}{\sqrt{2\pi}}\left(-\int_{-\infty}^{0} e^{x}e^{-iwx}dx + \int_{0}^{\infty} e^{-x}e^{-iwx}dx\right)\\
&= \frac{1}{\sqrt{2\pi}}\left(-\int_{-\infty}^{0} e^{(1-iw)x}dx + \int_{0}^{\infty} e^{-(1+iw)x}dx\right)\\
&= \frac{1}{\sqrt{2\pi}}\left(-\left[\frac{1}{1-iw}e^{(1-iw)x}\right]_{-\infty}^{0} - \left[\frac{1}{1+iw}e^{-(1+iw)x}\right]_{0}^{\infty}\right)\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1+iw} - \frac{1}{1-iw}\right)\\
&= -\frac{\sqrt{2}iw}{\sqrt{\pi}(1+w^2)}.
\end{aligned}
$$

Dette gir

$$
\begin{aligned}
f(x) &= \mathscr{F}^{-1}\left(-\frac{\sqrt{2}iw}{\sqrt{\pi}(1+w^2)}\right)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} -\frac{\sqrt{2}iwe^{iwx}}{\sqrt{\pi}(1+w^2)}dw\\
\Rightarrow f(1) = e^{-1} &= -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{iwe^{iw}}{(1+w^2)}dw.
\end{aligned}
$$

(Her kan vi også skrive $f(x) = \frac{2}{\pi}\int_0^\infty \frac{w \sin wx}{1+w^2} dw$, sidan f er odde.) Sidan

$$I := \int_0^\infty \frac{w \sin w}{1+w^2} dw = \frac{1}{2}\int_{-\infty}^\infty \frac{w \sin w}{1+w^2} dw$$
$$= -\frac{1}{2}\text{Re}\int_{-\infty}^\infty \frac{iwe^{iw}}{1+w^2} dw,$$

er

$$I = -\frac{1}{2}\text{Re}\left(-\frac{\pi}{e}\right) = \frac{\pi}{2e}.$$

I)

$$\begin{aligned}
\hat{f}(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \cos t e^{-t^2} e^{-iwt} dt \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{1}{2}\left(e^{it} + e^{-it}\right) e^{-t^2} e^{-iwt} dt \\
&= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^\infty e^{-t^2}\left(e^{-i(w-1)t} + e^{-(w+1)t}\right) dt \\
&= \frac{1}{2}(\mathscr{F}(e^{-t^2})(w-1) + \mathscr{F}(e^{-t^2})(w+1)) \\
&= \frac{1}{2\sqrt{2}}\left(e^{-(w-1)^2/4} + e^{-(w+1)^2/4}\right)
\end{aligned}$$

som også kan skrivast $\frac{1}{\sqrt{2}} e^{-(w^2+1)/4}\cosh(w/2)$. Her har vi brukt 9 i Tabell III side 536.

J) Vi bruker formelen

$$\mathscr{F}(f * g) = \sqrt{2\pi}\mathscr{F}(f)\mathscr{F}(g)$$

og får

$$\begin{aligned}
\mathscr{F}(h) = \mathscr{F}(e^{-x^2} * e^{-x^2}) &= \sqrt{2\pi}\mathscr{F}(e^{-x^2})\mathscr{F}(e^{-x^2}) \\
&= \sqrt{2\pi}\left(\frac{1}{\sqrt{2}} e^{-w^2/4}\right)^2 \\
&= \sqrt{\frac{\pi}{2}} e^{-w^2/2}.
\end{aligned}$$

Vi ser at med $a = \frac{1}{2}$ i formel 9, Table III, har vi

$$\mathscr{F}(e^{-x^2/2}) = e^{-w^2/2} = \sqrt{\frac{2}{\pi}}\mathscr{F}(h)$$
$$\Rightarrow h(x) = \sqrt{\frac{\pi}{2}} e^{-x^2/2}.$$

K) Vi vil ta Fouriertransform av begge sider, og vi byrjar med å rekne ut følgande:

$$\begin{aligned}
\mathscr{F}(e^{-a|x|}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{-iwx} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(a-iw)x} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(a+iw)x} dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a-iw} + \frac{1}{a+iw} \right) \\
&= \frac{2a}{\sqrt{2\pi}(a^2+w^2)}.
\end{aligned}$$

Integralet i likninga er konvolusjonen $f * e^{-3|x|}$, så, sidan $\mathscr{F}(f*g) = \sqrt{2\pi}\mathscr{F}(f)\mathscr{F}(g)$, kan vi Fouriertransformere begge sider av likninga og få

$$\begin{aligned}
\hat{f}(w) - \sqrt{2\pi} \frac{6}{\sqrt{2\pi}(9+w^2)} \hat{f}(w) &= \frac{6}{\sqrt{2\pi}(9+w^2)} \\
\Rightarrow \frac{3+w^2}{9+w^2} \hat{f}(w) &= \frac{6}{\sqrt{2\pi}(9+w^2)} \\
\Rightarrow \hat{f}(w) &= \frac{6}{\sqrt{2\pi}(3+w^2)}.
\end{aligned}$$

Med $a = \sqrt{3}$ over gir dette

$$f(x) = \sqrt{3} e^{-\sqrt{3}|x|}.$$

La)

$$\begin{aligned}
\hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx} dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{-e^{-iwx}}{iw} \right]_{-1}^{1} \\
&= \frac{-e^{-iw} + e^{iw}}{iw\sqrt{2\pi}} \\
&= \frac{2i \sin w}{iw\sqrt{2\pi}} \\
&= \sqrt{\frac{2}{\pi}} \frac{\sin w}{w}
\end{aligned}$$

$$
\begin{aligned}
\hat{g}(w) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(1+iw)x} dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{-e^{-(1+iw)x}}{1+iw} \right]_0^\infty \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{1+iw} \\
&= \frac{1}{\sqrt{2\pi}} \frac{1-iw}{1+w^2}
\end{aligned}
$$

Lb) Vi har $\mathscr{F}(h) = \mathscr{F}(f * g) = \sqrt{2\pi}\mathscr{F}(f)\mathscr{F}(g)$. Vi tek invers Fouriertransform og får

$$
\begin{aligned}
h(x) = \mathscr{F}^{-1}(\sqrt{2\pi}\hat{f}\hat{g}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \sqrt{2\pi}\hat{f}(w)\hat{g}(w)e^{iwx} dw \\
&= \frac{1}{\pi} \int_{-\infty}^\infty \frac{(1-iw)\sin w}{w(1+w^2)} e^{iwx} dw.
\end{aligned}
$$

Set vi $x = 0$, får vi

$$
\begin{aligned}
\frac{1}{\pi} \int_{-\infty}^\infty \frac{(1-iw)\sin w}{w(1+w^2)} dw &= h(0) \\
&= \int_{-\infty}^\infty f(-p)g(p)dp \\
&= \int_{-1}^1 g(p)dp \\
&= \int_0^1 e^{-p} dp \\
&= 1 - e^{-1}
\end{aligned}
$$

Integralet vi skal finne er realdelen av integralet på venstresida. Sidan høgresida er reell, får vi

$$
\frac{1}{\pi} \int_{-\infty}^\infty \frac{\sin w}{w(1+w^2)} dw = 1 - e^{-1},
$$

altså

$$
\int_{-\infty}^\infty \frac{\sin w}{w(1+w^2)} dw = \pi(1 - e^{-1}).
$$

M) Vi har $\sin w = i(e^{iw} - e^{-iw})/2$. Sidan f er den same funksjonen som i L), har vi

$$
\begin{aligned}
\hat{f}(w) &= \sqrt{\frac{2}{\pi}} \frac{\sin w}{w} \\
&= \frac{i}{\sqrt{2\pi}} \frac{e^{iw} - e^{-iw}}{w}.
\end{aligned}
$$

Så finn vi $\mathscr{F}(f * f)$:

$$\begin{aligned}\mathscr{F}(f * f) = \sqrt{2\pi}\hat{f} \cdot \hat{f} &= \frac{-1}{\sqrt{2\pi}} \left(\frac{e^{iw} - e^{-iw}}{w} \right)^2 \\ &= \frac{-1}{\sqrt{2\pi}} \frac{e^{2iw} - 2 + e^{-2iw}}{w^2}.\end{aligned}$$

Sidan $f * f$ er kontinuerleg, gir invers Fouriertransform

$$(f * f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{-1}{\sqrt{2\pi}} \frac{e^{2iw} - 2 + e^{-2iw}}{w^2} e^{iwx} dw.$$

Set vi $x = 3$, får vi

$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{5iw} - 2e^{3iw} + e^{iw}}{w^2} dw = (f * f)(3) = 0,$$

sidan $(f * f)(3) = \int_{-\infty}^{\infty} f(p)f(3-p)dp = \int_{-1}^{1} f(3-p)dp = \int_{2}^{4} f(t)dt = 0$. Integralet vi er ute etter er, sidan $e^{iaw} = \cos aw + i \sin aw$, realdelen av integralet på venstresida. Dermed er

$$\int_{-\infty}^{\infty} \frac{\cos 5w - 2\cos 3w + \cos w}{w^2} dw = 0.$$