Løysingsforslag for TMA4120, Øving 4

September 17, 2016

11.3.15) Vi finn først Fourierrekka til $r(t)$. r er odde, så $a_n = 0$ for alle n.

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(t\pi^2 - t^3)\sin nt dt \\
&= \frac{1}{\pi}\left(\left[-\frac{1}{n}(t\pi^2 - t^3)\cos nt\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{1}{n}(\pi^2 - 3t^2)\cos nt dt\right) \\
&= \int_{-\pi}^{\pi}\left(\frac{\pi}{n} - \frac{1}{n\pi}3t^2\right)\cos nt dt \\
&= \left[\frac{\pi}{n^2}\sin nt\right]_{-\pi}^{\pi} - \frac{3}{n\pi}\left(\left[\frac{1}{n}t^2\sin nt\right]_{-\pi}^{\pi} - \frac{2}{n}\int_{-\pi}^{\pi} t\sin nt \mathrm{d}t\right) \\
&= -\frac{3}{n\pi}\left(0 + \frac{2}{n^2}\left[t\cos nt\right]_{-\pi}^{\pi} - \frac{2}{n^2}\int_{-\pi}^{\pi}\cos nt \mathrm{d}t\right) \\
&= -\frac{3}{n\pi}\left(\frac{4}{n^2}\cdot\pi\cos n\pi + 0\right) \\
&= \frac{12(-1)^{n+1}}{n^3}
\end{aligned}$$

Med andre ord, $r(t) = \sum_1^\infty \frac{12(-1)^{n+1}}{n^3}\sin nt$. Så ser vi etter ei løysing $y_n = A_n\cos nt + B_n\sin nt$ av

$$y'' + cy' + y = \frac{12(-1)^{n+1}}{n^3}\sin nt$$

for $n \geq 1$. Vi får

$$\begin{aligned}
-n^2A_n + cnB_n + A_n &= 0 \\
\Rightarrow A_n &= -\frac{cn}{1-n^2}B_n \\
-n^2B_n - cnA_n + B_n &= \frac{12(-1)^{n+1}}{n^3} \\
\Rightarrow A_n &= \frac{1-n^2}{cn}B_n + \frac{12(-1)^n}{cn^4}
\end{aligned}$$

som gir

$$
\begin{aligned}
B_n &= \left(\frac{cn}{1-n^2} + \frac{1-n^2}{cn}\right)^{-1} \frac{12(-1)^{n+1}}{cn^4} \\
&= \left(\frac{c^2n^2 + 1 - 2n^2 + n^4}{cn(1-n^2)}\right)^{-1} \frac{12(-1)^{n+1}}{cn^4} \\
&= \frac{12(1-n^2)(-1)^{n+1}}{(n^4 + (c^2-2)n^2 + 1)n^3}
\end{aligned}
$$

og

$$
A_n = \frac{12c(-1)^n}{(n^4 + (c^2-2)n^2 + 1)n^2}.
$$

Løysinga er dermed

$$
y = \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx),
$$

der A_n og B_n er gitt over.

11.3.19) La $Q'(t) = I(t)$. Vi vil løyse

$$
\begin{aligned}
Q''' + 10Q'' + 10Q' &= E'(t) \\
\Rightarrow Q'' + 10Q' + 10Q &= E(t)
\end{aligned}
$$

I 11.3.15) fann vi ut at Fourierkoeffisientane til $\frac{1}{200}E(t)$ er $a_n = 0$ og $b_n = \frac{12(-1)^{n+1}}{n^3}$. Vi bruker same metode som sist og løyser følgande likningsystem for $n \geq 1$:

$$
\begin{aligned}
-n^2A_n + 10nB_n + 10A_n &= 0 \\
\Rightarrow A_n &= -\frac{10n}{10-n^2}B_n \\
-n^2B_n - 10nA_n + 10B_n &= \frac{2400(-1)^{n+1}}{n^3} \\
\Rightarrow A_n &= \frac{10-n^2}{10n}B_n + \frac{240(-1)^n}{n^4} \\
\Rightarrow B_n &= \left(\frac{10n}{10-n^2} + \frac{10-n^2}{10n}\right)^{-1} \frac{240(-1)^{n+1}}{n^4} \\
&= \left(\frac{10^2n^2 + (10-n^2)^2}{10n(10-n^2)}\right)^{-1} \frac{240(-1)^{n+1}}{n^4} \\
&= \frac{2400(10-n^2)(-1)^{n+1}}{(100n^2 + (10-n^2)^2)n^3}
\end{aligned}
$$

og dermed

$$A_n = \frac{24000(-1)^n}{(100n^2 + (10-n^2)^2)n^2}.$$

Dette gir oss Fourierkoeffisientane til Q, men det vi er ute etter er I. Derivasjon gir oss

$$\begin{aligned} I(t) &= \left(\sum_{n=1}^{\infty}\left(\frac{24000(-1)^n}{(100n^2+(10-n^2)^2)n^2}\cos nt + \frac{2400(10-n^2)(-1)^{n+1}}{(100n^2+(10-n^2)^2)n^3}\sin nt\right)\right)'(t) \\ &= \sum_{n=1}^{\infty}\left(\frac{24000(-1)^{n+1}}{(100n^2+(10-n^2)^2)n}\sin nt + \frac{2400(10-n^2)(-1)^{n+1}}{(100n^2+(10-n^2)^2)n^2}\cos nt\right). \end{aligned}$$

Utg. 9, 11.4.9) Bruker Euler-formelen

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

der

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\mathrm{dx}.$$

For $n \neq 0$ får vi:

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\mathrm{dx} \\ &= \frac{1}{2\pi}\left(\left[\frac{1}{-in}xe^{-inx}\right]_{-\pi}^{\pi} - \frac{1}{-in}\int_{-\pi}^{\pi} e^{-inx}\mathrm{dx}\right) \\ &= \frac{1}{2\pi}\left(\frac{\pi i}{n}\left(e^{-in\pi} + e^{n\pi}\right) - \frac{1}{(-in)^2}\left(e^{-in\pi} - e^{in\pi}\right)\right) \\ &= \frac{1}{2\pi}\left(\frac{\pi i}{n}2\cos n\pi + \frac{1}{n^2}(-2\sin n\pi)\right) \\ &= \frac{1}{2\pi}\frac{2\pi i}{n}(-1)^n + 0 \\ &= i\frac{(-1)^n}{n} \end{aligned}$$

For $n = 0$:

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x \cdot 1\mathrm{dx} = 0$$

Så vi kan skrive

$$f(x) = i\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n} e^{-inx}$$

Utg. 9, 11.4.10) Generelt kan vi skrive ei kompleks Fourier-rekke (på intervallet $-\pi < x < \pi$) som

$$f(x) = c_0 + \sum_{n=1}^{\infty}(c_n e^{inx} + c_{-n}e^{-inx})$$

Her er $c_0 = 0$, $c_n = i\frac{(-1)^n}{n}$ og $c_{(-n)} = i\frac{(-1)^{-n}}{-n} = -i\frac{(-1)^n}{n}$, så dette blir

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty}\left(i\frac{(-1)^n}{n}e^{inx} - i\frac{(-1)^n}{n}e^{-inx}\right) \\ &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}i(e^{inx} - e^{-inx}) \\ &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}i\cdot 2i\sin nx \\ &= \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx \end{aligned}$$

Utg. 9, 11.4.13) Finn Fourierkoeffisientane:

$$\begin{aligned} c_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)dx \\ &= \pi, \\ c_n &= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} xe^{-inx}dx \\ &= \frac{1}{2\pi}\left[\frac{i}{n}xe^{-inx} + \frac{1}{n^2}e^{-inx}\right]_0^{2\pi} \\ &= \frac{1}{2\pi}\left(\frac{2\pi i}{n} + \frac{1}{n^2} - \frac{1}{n^2}\right) \\ &= \frac{i}{n}, \end{aligned}$$

for $n \neq 0$.

11.4.4) Oppgåve 11.1.14 (Øving 3)

$$\implies f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos(nx)$$

$$\implies F(x) = \frac{\pi^2}{3} + \sum_{n=1}^{N}\frac{4}{n^2}(-1)^n\cos(nx) \qquad \text{for } N = 1, 2, ...$$

Minimum square error:

$$E^* = \int_{-\pi}^{\pi} f(x)^2 dx - \pi \left[2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \right]$$

Her: $f(x) = x^2$, $a_0 = \frac{\pi^2}{3}$, $a_1 = -4$, $a_2 = 1$, $a_3 = \frac{-4}{9}$, $a_4 = \frac{1}{4}$, $a_5 = -\frac{4}{25}$, $b_n = 0$ $n = 1, 2, 3, 4, 5$.

$$\begin{aligned}
\implies N = 1 \quad E^* &= \frac{2\pi^5}{5} - \pi \left[\frac{2\pi^4}{9} + 16 \right] \approx 4.14 \\
N = 2 \quad E^* &= \frac{2\pi^5}{5} - \pi \left[\frac{2\pi^4}{9} + 16 + 1 \right] \approx 1 \\
N = 3 \quad E^* &= \frac{2\pi^5}{5} - \pi \left[\frac{2\pi^4}{9} + 16 + 1 + \frac{16}{81} \right] \approx 0.38 \\
N = 4 \quad E^* &= \frac{2\pi^5}{5} - \pi \left[\frac{2\pi^4}{9} + 16 + 1 + \frac{16}{81} + \frac{1}{16} \right] \approx 0.18 \\
N = 5 \quad E^* &= \frac{2\pi^5}{5} - \pi \left[\frac{2\pi^4}{9} + 16 + 1 + \frac{16}{81} + \frac{1}{16} + \frac{16}{625} \right] \approx 0.1
\end{aligned}$$

11.4.8) I Fourierrekka til f er $b_n = 0$, sidan f er like.

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin x dx \\
&= \frac{2}{\pi} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} \sin x \cos nx dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} \frac{-i}{2}(e^{ix} - e^{-ix}) \cdot \frac{1}{2}(e^{inx} + e^{-inx})dx \\
&= \frac{-i}{2\pi}\int_{0}^{\pi} (e^{(n+1)ix} + e^{-(n-1)ix} - e^{(n-1)ix} - e^{-(n+1)ix})dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} (\sin(n+1)x + \sin(-(n-1)x))dx \\
&= \frac{1}{\pi}\left[-\frac{1}{n+1}\cos(n+1)x + \frac{1}{n-1}\cos(n-1)x\right]_0^{\pi} \\
&= \frac{1}{\pi} \cdot \frac{2}{n^2-1}((-1)^{n+1} - 1) \\
&= \frac{2((-1)^{n+1} - 1)}{\pi(n^2-1)}.
\end{aligned}$$

Dette betyr at $a_n = 0$ for odde n, og $a_n = 4/(\pi(1-n^2))$ for like $n \geq 1$.

Minimum square error:

$$E^* = \int_{-\pi}^{\pi} \sin^2 x dx - \pi\left[\frac{8}{\pi^2} + \sum_{n=1}^{N} a_n^2\right]$$

Her er $a_0 = \frac{2}{\pi}$, $a_1 = a_3 = a_5 = 0$, $a_2 = -4/(3\pi)$, $a_4 = -4/(15\pi)$, og

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin^2 x dx &= \int_{-\pi}^{\pi} \frac{1}{2}(1 - \cos 2x)dx \\
&= \pi
\end{aligned}$$

$$
\begin{aligned}
\implies N = 1 \quad & E^* = \pi - \pi \left[\frac{8}{\pi^2}\right] \approx 0,60 \\
N = 2 \quad & E^* = \pi - \pi \left[\frac{8}{\pi^2} + \frac{16}{9\pi^2}\right] \approx 0,029 \\
N = 3 \quad & E^* = \pi - \pi \left[\frac{8}{\pi^2} + \frac{16}{9\pi^2}\right] \approx 0,029 \\
N = 4 \quad & E^* = \pi - \pi \left[\frac{8}{\pi^2} + \frac{16}{9\pi^2} + \frac{16}{225\pi^2}\right] \approx 0,0066 \\
N = 5 \quad & E^* = \pi - \pi \left[\frac{8}{\pi^2} + \frac{16}{9\pi^2} + \frac{16}{225\pi^2}\right] \approx 0,0066
\end{aligned}
$$

11.4.11) Eksempel 1 i 11.1 viser at for

$$
f(x) = \begin{cases} -\frac{\pi}{4}, & -\pi < x < 0, \\ \frac{\pi}{4}, & 0 < x < \pi \end{cases},
$$

så er Fourierkoeffisientane $a_n = 0$ for alle n, $b_n = 0$ for like n, og $b_n = 1/n$ for odde n. Parcevals identitet gir dermed

$$
\begin{aligned}
1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\pi^2}{4^2} dx \\
&= \frac{\pi^2}{8}
\end{aligned}
$$

11.R.15) Sidan alle cosinusledda er like og sinusledda er odde, må svaret vere to funksjonar g og h der g er like, h odde, og $g(x) + h(x) = f(x) = e^x$ for $-5 < x < 5$. Vi har

$$
\begin{aligned}
f(x) + f(-x) &= g(x) + h(x) + g(-x) + h(-x) \\
&= 2g(x),
\end{aligned}
$$

som gir $g(x) = (e^x + e^{-x})/2 = \cosh x$ og $h(x) = e^x - \cosh x = \sinh x$. Dermed er $\cosh x$ summen av cosinusledda, og $\sinh x$ summen av sinusledda.

11.R.17) Eksempel 1 i 11.1 gir ei løysing av oppgåva. Eventuelt kan ein bruke Eksempel 1 i 11.2, der vi får vite at

$$
f(x) = \begin{cases} 0, & -2 < x < -1 \\ k, & -1 < x < 1 \\ 0, & 1 < x < 2 \end{cases}
$$

har Fourierrekka

$$
f(x) = \frac{k}{2} + \frac{2k}{\pi} \left(\cos \frac{\pi}{2} x - \frac{1}{3} \cos \frac{3\pi}{2} x + \frac{1}{5} \cos \frac{5\pi}{2} x - \cdots \right).
$$

Med $x = 0$ og $k = \pi/2$ gir dette

$$\frac{\pi}{2} = \frac{\pi}{4} + \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right)$$
$$\Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}.$$

E) Sidan $f(x)$ er kontinuerleg (og tilstrekkeleg glatt) konvergerer Fourier-sinus-rekka mot $f(x)$ for alle $x \in [0, \pi]$. Vi har

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}, \sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}, \sin\frac{7\pi}{4} = -\frac{1}{\sqrt{2}}, \sin\frac{9\pi}{4} = \sin\frac{1\pi}{4} = \frac{1}{\sqrt{2}}, \cdots$$

Vel $x = \frac{\pi}{4}$ i Fourierrekka og får

$$\begin{aligned} f\left(\frac{\pi}{4}\right) &= \frac{3\pi^2}{16} \\ &= \frac{8}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)\pi/4}{(2m+1)^3} \\ &= \frac{8}{\pi}\frac{1}{\sqrt{2}} \left(\frac{1}{1^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} - \frac{1}{13^3} - \frac{1}{15^3} + \cdots\right). \end{aligned}$$

Altså er

$$\frac{1}{1^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} - \frac{1}{13^3} - \frac{1}{15^3} + \cdots = \frac{3\sqrt{2}\pi^3}{128}$$

F a) Vi finn koeffisientane til den komplekse Fourierrekka $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$:

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-inx} dx \\ &= \frac{1}{2\pi} \left[\frac{1}{1-in} e^{(1-in)x}\right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi(1-in)} (e^{(1-in)\pi} - e^{-(1-in)\pi}) \\ &= \frac{1}{2\pi(1-in)} (e^{\pi} e^{-in\pi} - e^{-\pi} e^{in\pi}) \\ &= \frac{(-1)^n (e^{\pi} - e^{-\pi})}{2\pi(1-in)} \\ &= \frac{(-1)^n \sinh \pi}{\pi(1-in)} \end{aligned}$$

b) Set $x = 0$ og får

$$\begin{aligned}
1 = f(0) &= \sum_{n=-\infty}^{\infty} c_n \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh \pi}{\pi(1-in)} \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1+in) \sinh \pi}{\pi(1+n^2)}.
\end{aligned}$$

Vi ser at dei imaginære ledda i summen forsvinn (som venta, sidan venstresida er reell). Det gir

$$\begin{aligned}
1 &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh \pi}{\pi(1+n^2)} \\
\Rightarrow \frac{\pi}{\sinh \pi} &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1+n^2} \\
\Rightarrow \frac{\pi}{2\sinh \pi} &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} \\
\Rightarrow \frac{\pi}{2\sinh \pi} &= \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}.
\end{aligned}$$

Ettersom $f(x)$ har eit sprang i $x = \pi$ får vi, sidan $e^{in\pi} = (-1)^n$, at

$$\begin{aligned}
\frac{1}{2}(f(\pi^+) + f(\pi^-)) = \cosh \pi &= \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1-in}(-1)^n \\
&= \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} \\
&= \frac{\sinh \pi}{\pi} \left(1 + 1 + 2\sum_{n=2}^{\infty} \frac{1}{1+n^2} \right) \\
\Rightarrow \sum_{n=2}^{\infty} \frac{1}{1+n^2} &= \frac{\pi \cosh \pi}{2 \sinh \pi} - 1 \\
&= \frac{\pi}{2\tanh \pi} - 1
\end{aligned}$$

G) La $F(x)$ vere summen av Fourierrekka for $f(x)$. Då er $F(x) = f(x)$ for alle x der f er kontinuerleg. Sidan $f(x)$ er kontinuerleg for alle x, også i punkta $x = n\pi$, er $F(x) = f(x)$ for alle x, det vil seie,

$$f(x) = x^4 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{8(-1)^n(\pi^2 n^2 - 6)}{n^4} \cos nx \text{ for } -\pi \le x \le \pi$$

og derfor

$$f(\pi) = \pi^4 = \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{8(-1)^n(\pi^2 n^2 - 6)}{n^4} \cos n\pi = \frac{\pi^4}{5} + 8 \sum_{n=1}^{\infty} \frac{\pi^2 n^2 - 6}{n^4}$$

og dermed

$$\sum_{n=1}^{\infty} \frac{\pi^2 n^2 - 6}{n^4} = \frac{1}{8} \left(1 - \frac{1}{5}\right) \pi^4 = \frac{\pi^4}{10}.$$

La S vere summen av den neste rekka. Ho kan skrivast som $\frac{1}{64} \sum_{n=1}^{\infty} a_n^2$. For dei som hugsar Parcevals formel

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2$$

kan oppgåva løysast slik:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^8 dx = 2 \left(\frac{\pi^4}{5}\right)^2 + 64S$$

der vestre side er lik $2\pi^8/9$, slik at $S = (2\pi^8/9 - 2(\pi^4/5)^2)/64 = \pi^8/450$.

Vi andre, vanlege dødelege kan til dømes tenke slik: $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$. Altså er

$$\begin{aligned} f(x)^2 &= \left(a_0 + \sum_{n=1}^{\infty} a_n \cos nx\right) \left(a_0 + \sum_{n=1}^{\infty} a_n \cos nx\right) \\ &= a_0^2 + \sum_{n=1}^{\infty} a_n^2 \cos^2 nx + \text{ masse ledd med } \cos nx \cos mx \text{ der } 0 \le m < n. \end{aligned}$$

For å kvitte oss med alle ledda med $\cos nx \cos mx$, kan vi integrere over ei periode:

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)^2 dx &= \int_{-\pi}^{\pi} a_0^2 dx + \sum_{n=1}^{\infty} a_n^2 \int_{-\pi}^{\pi} \cos^2 nx dx \\ &\quad + \text{ masse ledd med } \int_{-\pi}^{\pi} \cos mx \cos nx \text{ der } m \neq n. \end{aligned}$$

Heile Fourierrekketeorien er basert på at

$$\int_{-\pi}^{\pi} \cos^2 nx dx = \pi \text{ og } \int_{-\pi}^{\pi} \cos mx \cos nx dx = 0 \text{ for } m \neq n.$$

(Det kan også lett reknast ut.) Dermed får vi

$$\int_{-\pi}^{\pi} f(x)^2 dx = a_0^2 \cdot 2\pi + \sum_{n=1}^{\infty} a_n^2 \pi + 0 = \pi \left(2a_0^2 + \sum_{n=1}^{\infty} a_n^2\right)$$

og ein finn summen S som over.