

Løysingsforslag for TMA4120, Øving 4

September 29, 2016

11.3.15) Vi finn først Fourierrekka til $r(t)$. r er odde, så $a_n = 0$ for alle n .

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} (t\pi^2 - t^3) \sin ntdt \\ &= \frac{1}{\pi} \left(\left[-\frac{1}{n}(t\pi^2 - t^3) \cos nt \right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{1}{n}(\pi^2 - 3t^2) \cos ntdt \right) \\ &= \int_{-\pi}^{\pi} \left(\frac{\pi}{n} - \frac{1}{n\pi} 3t^2 \right) \cos ntdt \\ &= \left[\frac{\pi}{n^2} \sin nt \right]_{-\pi}^{\pi} - \frac{3}{n\pi} \left(\left[\frac{1}{n} t^2 \sin nt \right]_{-\pi}^{\pi} - \frac{2}{n} \int_{-\pi}^{\pi} t \sin ntdt \right) \\ &= -\frac{3}{n\pi} \left(0 + \frac{2}{n^2} [t \cos nt]_{-\pi}^{\pi} - \frac{2}{n^2} \int_{-\pi}^{\pi} \cos ntdt \right) \\ &= -\frac{3}{n\pi} \left(\frac{4}{n^2} \cdot \pi \cos n\pi + 0 \right) \\ &= \frac{12(-1)^{n+1}}{n^3} \end{aligned}$$

Med andre ord, $r(t) = \sum_1^{\infty} \frac{12(-1)^{n+1}}{n^3} \sin nt$. Så ser vi etter ei løysing $y_n = A_n \cos nt + B_n \sin nt$ av

$$y'' + cy' + y = \frac{12(-1)^{n+1}}{n^3} \sin nt$$

for $n \geq 1$. Vi får

$$\begin{aligned} -n^2 A_n + cnB_n + A_n &= 0 \\ \Rightarrow A_n &= -\frac{cn}{1-n^2} B_n \\ -n^2 B_n - cnA_n + B_n &= \frac{12(-1)^{n+1}}{n^3} \\ \Rightarrow A_n &= \frac{1-n^2}{cn} B_n + \frac{12(-1)^n}{cn^4} \end{aligned}$$

som gir

$$\begin{aligned} B_n &= \left(\frac{cn}{1-n^2} + \frac{1-n^2}{cn} \right)^{-1} \frac{12(-1)^{n+1}}{cn^4} \\ &= \left(\frac{c^2n^2 + 1 - 2n^2 + n^4}{cn(1-n^2)} \right)^{-1} \frac{12(-1)^{n+1}}{cn^4} \\ &= \frac{12(1-n^2)(-1)^{n+1}}{(n^4 + (c^2 - 2)n^2 + 1)n^3} \end{aligned}$$

og

$$A_n = \frac{12c(-1)^n}{(n^4 + (c^2 - 2)n^2 + 1)n^2}.$$

Løysinga er dermed

$$y = \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx),$$

der A_n og B_n er gitt over.

11.3.19) La $Q'(t) = I(t)$. Vi vil løyse

$$\begin{aligned} Q''' + 10Q'' + 10Q' &= E'(t) \\ \Rightarrow Q'' + 10Q' + 10Q &= E(t) \end{aligned}$$

I 11.3.15) fann vi ut at Fourierkoeffisientane til $\frac{1}{200}E(t)$ er $a_n = 0$ og $b_n = \frac{12(-1)^{n+1}}{n^3}$. Vi bruker same metode som sist og løyser følgande likningsystem for $n \geq 1$:

$$\begin{aligned} -n^2A_n + 10nB_n + 10A_n &= 0 \\ \Rightarrow A_n &= -\frac{10n}{10-n^2}B_n \\ -n^2B_n - 10nA_n + 10B_n &= \frac{2400(-1)^{n+1}}{n^3} \\ \Rightarrow A_n &= \frac{10-n^2}{10n}B_n + \frac{240(-1)^n}{n^4} \\ \Rightarrow B_n &= \left(\frac{10n}{10-n^2} + \frac{10-n^2}{10n} \right)^{-1} \frac{240(-1)^{n+1}}{n^4} \\ &= \left(\frac{10^2n^2 + (10-n^2)^2}{10n(10-n^2)} \right)^{-1} \frac{240(-1)^{n+1}}{n^4} \\ &= \frac{2400(10-n^2)(-1)^{n+1}}{(100n^2 + (10-n^2)^2)n^3} \end{aligned}$$

og dermed

$$A_n = \frac{24000(-1)^n}{(100n^2 + (10 - n^2)^2)n^2}.$$

Dette gir oss Fourierkoeffisientane til Q , men det vi er ute etter er I . Derivasjon gir oss

$$\begin{aligned} I(t) &= \left(\sum_{n=1}^{\infty} \left(\frac{24000(-1)^n}{(100n^2 + (10 - n^2)^2)n^2} \cos nt + \frac{2400(10 - n^2)(-1)^{n+1}}{(100n^2 + (10 - n^2)^2)n^3} \sin nt \right) \right)' (t) \\ &= \sum_{n=1}^{\infty} \left(\frac{24000(-1)^{n+1}}{(100n^2 + (10 - n^2)^2)n} \sin nt + \frac{2400(10 - n^2)(-1)^{n+1}}{(100n^2 + (10 - n^2)^2)n^2} \cos nt \right). \end{aligned}$$

Utg. 9, 11.4.9) Bruker Euler-formelen

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

der

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

For $n \neq 0$ får vi:

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} dx \\ &= \frac{1}{2\pi} \left(\left[\frac{1}{-in} x e^{-inx} \right]_{-\pi}^{\pi} - \frac{1}{-in} \int_{-\pi}^{\pi} e^{-inx} dx \right) \\ &= \frac{1}{2\pi} \left(\frac{\pi i}{n} (e^{-in\pi} + e^{n\pi}) - \frac{1}{(-in)^2} (e^{-in\pi} - e^{in\pi}) \right) \\ &= \frac{1}{2\pi} \left(\frac{\pi i}{n} 2 \cos n\pi + \frac{1}{n^2} (-2 \sin n\pi) \right) \\ &= \frac{1}{2\pi} \frac{2\pi i}{n} (-1)^n + 0 \\ &= i \frac{(-1)^n}{n} \end{aligned}$$

For $n = 0$:

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \cdot 1 dx = 0$$

Så vi kan skrive

$$f(x) = i \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^n}{n} e^{-inx}$$

Utg. 9, 11.4.10) Generelt kan vi skrive ei kompleks Fourier-rekke (på intervallet $-\pi < x < \pi$) som

$$f(x) = c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + c_{-n} e^{-inx})$$

Her er $c_0 = 0$, $c_n = i \frac{(-1)^n}{n}$ og $c_{(-n)} = i \frac{(-1)^{-n}}{-n} = -i \frac{(-1)^n}{n}$, så dette blir

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \left(i \frac{(-1)^n}{n} e^{inx} - i \frac{(-1)^n}{n} e^{-inx} \right) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} i (e^{inx} - e^{-inx}) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} i \cdot 2i \sin nx \\ &= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx \end{aligned}$$

Utg. 9, 11.4.13) Finn Fourierkoeffisientane:

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(x) dx \\ &= \pi, \\ c_n &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} dx \\ &= \frac{1}{2\pi} \left[\frac{i}{n} x e^{-inx} + \frac{1}{n^2} e^{-inx} \right]_0^{2\pi} \\ &= \frac{1}{2\pi} \left(\frac{2\pi i}{n} + \frac{1}{n^2} - \frac{1}{n^2} \right) \\ &= \frac{i}{n}, \end{aligned}$$

for $n \neq 0$.

11.4.4) Oppgave 11.1.14 (Øving 3)

$$\begin{aligned} \implies f(x) &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos(nx) \\ \implies F(x) &= \frac{\pi^2}{3} + \sum_{n=1}^N \frac{4}{n^2} (-1)^n \cos(nx) \quad \text{for } N = 1, 2, \dots \end{aligned}$$

Minimum square error:

$$E^* = \int_{-\pi}^{\pi} f(x)^2 dx - \pi \left[2a_0^2 + \sum_{n=1}^N (a_n^2 + b_n^2) \right]$$

Her: $f(x) = x^2$, $a_0 = \frac{\pi^2}{3}$, $a_1 = -4$, $a_2 = 1$, $a_3 = \frac{-4}{9}$, $a_4 = \frac{1}{4}$, $a_5 = -\frac{4}{25}$, $b_n = 0$
 $n = 1, 2, 3, 4, 5$.

$$\begin{aligned} \Rightarrow N = 1 \quad E^* &= \frac{2\pi^5}{5} - \pi \left[\frac{2\pi^4}{9} + 16 \right] \approx 4.14 \\ N = 2 \quad E^* &= \frac{2\pi^5}{5} - \pi \left[\frac{2\pi^4}{9} + 16 + 1 \right] \approx 1 \\ N = 3 \quad E^* &= \frac{2\pi^5}{5} - \pi \left[\frac{2\pi^4}{9} + 16 + 1 + \frac{16}{81} \right] \approx 0.38 \\ N = 4 \quad E^* &= \frac{2\pi^5}{5} - \pi \left[\frac{2\pi^4}{9} + 16 + 1 + \frac{16}{81} + \frac{1}{16} \right] \approx 0.18 \\ N = 5 \quad E^* &= \frac{2\pi^5}{5} - \pi \left[\frac{2\pi^4}{9} + 16 + 1 + \frac{16}{81} + \frac{1}{16} + \frac{16}{625} \right] \approx 0.1 \end{aligned}$$

11.4.8) I Fourierrekka til f er $b_n = 0$, sidan f er like.

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{\pi} \int_0^{\pi} \sin x dx \\ &= \frac{2}{\pi} \end{aligned}$$

Finne a_n for $n \geq 2$:

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\
 &= \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx dx \\
 &= \frac{2}{\pi} \int_0^{\pi} \frac{-i}{2} (e^{ix} - e^{-ix}) \cdot \frac{1}{2} (e^{inx} + e^{-inx}) dx \\
 &= \frac{-i}{2\pi} \int_0^{\pi} (e^{(n+1)ix} + e^{-(n-1)ix} - e^{(n-1)ix} - e^{-(n+1)ix}) dx \\
 &= \frac{1}{\pi} \int_0^{\pi} (\sin(n+1)x + \sin(-(n-1)x)) dx \\
 &= \frac{1}{\pi} \left[-\frac{1}{n+1} \cos(n+1)x + \frac{1}{n-1} \cos(n-1)x \right]_0^{\pi} \\
 &= \frac{1}{\pi} \cdot \frac{2}{n^2-1} ((-1)^{n+1} - 1) \\
 &= \frac{2((-1)^{n+1} - 1)}{\pi(n^2-1)}.
 \end{aligned}$$

For å finne a_1 gjer vi same utrekninga fram til femte linje, der $\sin(-(n-1)) = 0$, så vi får kun $\sin(2x)$ i integranden. Integralet blir dermed 0, så $a_1 = 0$. Dette betyr at $a_n = 0$ for odde n , og $a_n = 4/(\pi(1-n^2))$ for like $n \geq 1$.

Minimum square error:

$$E^* = \int_{-\pi}^{\pi} \sin^2 x dx - \pi \left[\frac{8}{\pi^2} + \sum_{n=1}^N a_n^2 \right]$$

Her er $a_0 = \frac{2}{\pi}$, $a_1 = a_3 = a_5 = 0$, $a_2 = -4/(3\pi)$, $a_4 = -4/(15\pi)$, og

$$\begin{aligned}
 \int_{-\pi}^{\pi} \sin^2 x dx &= \int_{-\pi}^{\pi} \frac{1}{2} (1 - \cos 2x) dx \\
 &= \pi
 \end{aligned}$$

$$\begin{aligned} \implies N = 1 \quad E^* &= \pi - \pi \left[\frac{8}{\pi^2} \right] \approx 0,60 \\ N = 2 \quad E^* &= \pi - \pi \left[\frac{8}{\pi^2} + \frac{16}{9\pi^2} \right] \approx 0,029 \\ N = 3 \quad E^* &= \pi - \pi \left[\frac{8}{\pi^2} + \frac{16}{9\pi^2} \right] \approx 0,029 \\ N = 4 \quad E^* &= \pi - \pi \left[\frac{8}{\pi^2} + \frac{16}{9\pi^2} + \frac{16}{225\pi^2} \right] \approx 0,0066 \\ N = 5 \quad E^* &= \pi - \pi \left[\frac{8}{\pi^2} + \frac{16}{9\pi^2} + \frac{16}{225\pi^2} \right] \approx 0,0066 \end{aligned}$$

11.4.11) Eksempel 1 i 11.1 viser at for

$$f(x) = \begin{cases} -\frac{\pi}{4}, & -\pi < x < 0, \\ \frac{\pi}{4}, & 0 < x < \pi \end{cases},$$

så er Fourierkoeffisientane $a_n = 0$ for alle n , $b_n = 0$ for like n , og $b_n = 1/n$ for odde n . Parsevals identitet gir dermed

$$\begin{aligned} 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\pi^2}{4^2} dx \\ &= \frac{\pi^2}{8} \end{aligned}$$

11.R.15) Sidan alle cosinusledda er like og sinusledda er odde, må svaret vere to funksjonar g og h der g er like, h odde, og $g(x) + h(x) = f(x) = e^x$ for $-5 < x < 5$. Vi har

$$\begin{aligned} f(x) + f(-x) &= g(x) + h(x) + g(-x) + h(-x) \\ &= 2g(x), \end{aligned}$$

som gir $g(x) = (e^x + e^{-x})/2 = \cosh x$ og $h(x) = e^x - \cosh x = \sinh x$. Dermed er $\cosh x$ summen av cosinusledda, og $\sinh x$ summen av sinusledda.

11.R.17) Eksempel 1 i 11.1 gir ei løysing av oppgåva. Eventuelt kan ein bruke Eksempel 1 i 11.2, der vi får vite at

$$f(x) = \begin{cases} 0, & -2 < x < -1 \\ k, & -1 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$$

har Fourierrekka

$$f(x) = \frac{k}{2} + \frac{2k}{\pi} \left(\cos \frac{\pi}{2} x - \frac{1}{3} \cos \frac{3\pi}{2} x + \frac{1}{5} \cos \frac{5\pi}{2} x - \dots \right).$$

Med $x = 0$ og $k = \pi/2$ gir dette

$$\begin{aligned}\frac{\pi}{2} &= \frac{\pi}{4} + \left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right) \\ \Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \dots &= \frac{\pi}{4}.\end{aligned}$$

E) Sidan $f(x)$ er kontinuerleg (og tilstrekkeleg glatt) konvergerer Fourier-sinus-rekka mot $f(x)$ for alle $x \in [0, \pi]$. Vi har

$$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}, \sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}, \sin \frac{5\pi}{4} = -\frac{1}{\sqrt{2}}, \sin \frac{7\pi}{4} = -\frac{1}{\sqrt{2}}, \sin \frac{9\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}, \dots$$

Vel $x = \frac{\pi}{4}$ i Fourierrekka og får

$$\begin{aligned}f\left(\frac{\pi}{4}\right) &= \frac{3\pi^2}{16} \\ &= \frac{8}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)\pi/4}{(2m+1)^3} \\ &= \frac{8}{\pi} \frac{1}{\sqrt{2}} \left(\frac{1}{1^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} - \frac{1}{13^3} - \frac{1}{15^3} + \dots\right).\end{aligned}$$

Altså er

$$\frac{1}{1^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} - \frac{1}{13^3} - \frac{1}{15^3} + \dots = \frac{3\sqrt{2}\pi^3}{128}$$

F a) Vi finn koeffisientane til den komplekse Fourierrekka $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$:

$$\begin{aligned}c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-inx} dx \\ &= \frac{1}{2\pi} \left[\frac{1}{1-in} e^{(1-in)x} \right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi(1-in)} (e^{(1-in)\pi} - e^{-(1-in)\pi}) \\ &= \frac{1}{2\pi(1-in)} (e^{\pi} e^{-in\pi} - e^{-\pi} e^{in\pi}) \\ &= \frac{(-1)^n (e^{\pi} - e^{-\pi})}{2\pi(1-in)} \\ &= \frac{(-1)^n \sinh \pi}{\pi(1-in)}\end{aligned}$$

b) Set $x = 0$ og får

$$\begin{aligned} 1 = f(0) &= \sum_{n=-\infty}^{\infty} c_n \\ &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh \pi}{\pi(1 - in)} \\ &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1 + in) \sinh \pi}{\pi(1 + n^2)}. \end{aligned}$$

Vi ser at dei imaginære ledda i summen forsvinn (som venta, sidan venstresida er reell). Det gir

$$\begin{aligned} 1 &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh \pi}{\pi(1 + n^2)} \\ \Rightarrow \frac{\pi}{\sinh \pi} &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1 + n^2} \\ \Rightarrow \frac{\pi}{2 \sinh \pi} &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1 + n^2} \\ \Rightarrow \frac{\pi}{2 \sinh \pi} &= \sum_{n=2}^{\infty} \frac{(-1)^n}{1 + n^2}. \end{aligned}$$

Ettersom $f(x)$ har eit sprang i $x = \pi$ får vi, sidan $e^{in\pi} = (-1)^n$, at

$$\begin{aligned} \frac{1}{2}(f(\pi^+) + f(\pi^-)) &= \cosh \pi = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1 - in} (-1)^n \\ &= \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{1 + n^2} \\ &= \frac{\sinh \pi}{\pi} \left(1 + 1 + 2 \sum_{n=2}^{\infty} \frac{1}{1 + n^2} \right) \\ \Rightarrow \sum_{n=2}^{\infty} \frac{1}{1 + n^2} &= \frac{\pi \cosh \pi}{2 \sinh \pi} - 1 \\ &= \frac{\pi}{2 \tanh \pi} - 1 \end{aligned}$$

G) La $F(x)$ vere summen av Fourierrekka for $f(x)$. Då er $F(x) = f(x)$ for alle x der f er kontinuerleg. Sidan $f(x)$ er kontinuerleg for alle x , også i punkta $x = n\pi$, er $F(x) = f(x)$ for alle x , det vil seie,

$$f(x) = x^4 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{8(-1)^n(\pi^2 n^2 - 6)}{n^4} \cos nx \text{ for } -\pi \leq x \leq \pi$$

og derfor

$$f(\pi) = \pi^4 = \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{8(-1)^n(\pi^2 n^2 - 6)}{n^4} \cos n\pi = \frac{\pi^4}{5} + 8 \sum_{n=1}^{\infty} \frac{\pi^2 n^2 - 6}{n^4}$$

og dermed

$$\sum_{n=1}^{\infty} \frac{\pi^2 n^2 - 6}{n^4} = \frac{1}{8} \left(1 - \frac{1}{5}\right) \pi^4 = \frac{\pi^4}{10}.$$

La S vere summen av den neste rekka. Ho kan skrivast som $\frac{1}{64} \sum_{n=1}^{\infty} a_n^2$. For dei som hugsar Parsevals formel

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2$$

kan oppgåva løysast slik:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^8 dx = 2 \left(\frac{\pi^4}{5}\right)^2 + 64S$$

der vestre side er lik $2\pi^8/9$, slik at $S = (2\pi^8/9 - 2(\pi^4/5)^2)/64 = \pi^8/450$.

Vi andre, vanlege dødelege kan til dømes tenke slik: $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$. Altså er

$$\begin{aligned} f(x)^2 &= \left(a_0 + \sum_{n=1}^{\infty} a_n \cos nx\right) \left(a_0 + \sum_{n=1}^{\infty} a_n \cos nx\right) \\ &= a_0^2 + \sum_{n=1}^{\infty} a_n^2 \cos^2 nx + \text{masse ledd med } \cos nx \cos mx \text{ der } 0 \leq m < n. \end{aligned}$$

For å kvitte oss med alle ledda med $\cos nx \cos mx$, kan vi integrere over ei periode:

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)^2 dx &= \int_{-\pi}^{\pi} a_0^2 dx + \sum_{n=1}^{\infty} a_n^2 \int_{-\pi}^{\pi} \cos^2 nx dx \\ &\quad + \text{masse ledd med } \int_{-\pi}^{\pi} \cos mx \cos nx \text{ der } m \neq n. \end{aligned}$$

Heile Fourierrekke-teorien er basert på at

$$\int_{-\pi}^{\pi} \cos^2 nx dx = \pi \text{ og } \int_{-\pi}^{\pi} \cos mx \cos nx dx = 0 \text{ for } m \neq n.$$

(Det kan også lett reknast ut.) Dermed får vi

$$\int_{-\pi}^{\pi} f(x)^2 dx = a_0^2 \cdot 2\pi + \sum_{n=1}^{\infty} a_n^2 \pi + 0 = \pi \left(2a_0^2 + \sum_{n=1}^{\infty} a_n^2\right)$$

og ein finn summen S som over.