# Løysingsforslag for TMA4120, Øving 3

September 19, 2016

**11.1.14)**

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 \mathrm{d}x \\
&= \frac{1}{2\pi}\cdot\frac{2\pi^3}{3} \\
&= \frac{\pi^2}{3} \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx \mathrm{d}x \\
&= 0 \\
&\text{(Odde funksjon)} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \mathrm{d}x \\
&= \frac{1}{\pi}\left(\left[\frac{1}{n}x^2 \sin nx\right]_{-\pi}^{\pi} - \frac{2}{n}\int_{-\pi}^{\pi} x \sin nx \mathrm{dx}\right) \\
&= \frac{1}{\pi}\left(0 + \frac{2}{n^2}\left[x\cos nx\right]_{-\pi}^{\pi} - \frac{2}{n^2}\int_{-\pi}^{\pi} \cos nx \mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(\frac{4}{n^2}\cdot \pi\cos n\pi + 0\right) \\
&= \frac{4(-1)^n}{n^2}
\end{aligned}$$

$$x^2 = \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x + \dots) \qquad (\text{for } -\pi < x < \pi)$$

**11.1.19)** Det er klart at funksjonen i oppgåva er gitt av

$$f(x) = \begin{cases} 0 & \text{for } -\pi \le t < 0 \\ x & \text{for } 0 \le t < \pi \end{cases}$$

(og er periodisk elles). Vi reknar ut Fourier-koeffisientane:

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_0^{\pi} xdx = \frac{1}{4\pi}\pi^2 = \frac{\pi}{4} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_0^{\pi} x\cos nxdx = \frac{1}{\pi}\left[\frac{x}{n}\sin nx\right]_{x=0}^{\pi} - \frac{1}{n\pi}\int_0^{\pi}\sin nxdx \\
&= \frac{1}{n}\sin n\pi + \frac{1}{n^2\pi}\cos n\pi - \frac{1}{n^2\pi} \\
&= \frac{1}{n^2\pi}\cos n\pi - \frac{1}{n^2\pi} \\
&= \begin{cases} -\frac{2}{\pi n^2} & \text{for } n = 1,3,5,\ldots \\ 0 & \text{for } n = 2,4,6,\ldots \end{cases} \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = \frac{1}{\pi}\int_0^{\pi} x\sin nxdx = -\frac{1}{\pi}\left[\frac{x}{n}\cos nx\right]_{x=0}^{\pi} - \frac{1}{n\pi}\int_0^{\pi}\cos nxdx \\
&= -\frac{1}{n}\cos nx \\
&= \frac{(-1)^{n+1}}{n}.
\end{aligned}
$$

**11.1.21)**

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = 0, \qquad \text{(opplagt ut frå teikninga.)} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0}(-x-\pi)\cos nxdx + \int_0^{\pi}(-x+\pi)\cos nxdx\right)
\end{aligned}
$$

sub: $y = -x$ i det første integralet gir

$$a_n = 0.$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0}(-x-\pi)\sin nxdx + \int_0^{\pi}(-x+\pi)\sin nxdx\right)
\end{aligned}
$$

sub: $y = -x$ i det første integralet gir

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^\pi (-x+\pi)\sin nx dx \\
&= \frac{2}{\pi}\left(\Big|_0^\pi \frac{x\cos nx}{n} + \frac{1}{n}\int_0^\pi \cos nx dx + \pi\int_0^\pi \sin nx\right) \\
&= \frac{2}{\pi}\left(\frac{\pi\cos n\pi}{n} - \frac{1}{n^2}\Big|_0^\pi \sin nx - \frac{\pi}{n}\Big|_0^\pi \cos nx\right) \\
&= \frac{2}{\pi}\left(\frac{\pi}{n}\cos n\pi - 0 - \frac{\pi}{n}(\cos n\pi - 1)\right) \\
&= \frac{2}{n}.
\end{aligned}$$

Ein kan også integrere frå 0 til $2\pi$ og bruke $-x+\pi$, slik at ein slepp å jobbe med to integral. Uansett får ein Fourierrekka

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}.$$

**11.2.11)** Frå 11.1.14 har vi

$$x^2 = \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x + \ldots \qquad (\text{for } -\pi < x < \pi).$$

La $v = x/\pi$. Då får vi

$$\begin{aligned}
\pi^2 v^2 &= \frac{\pi^2}{3} - 4\cos \pi v + \cos 2\pi v - \frac{4}{9}\cos 3\pi v + \ldots \qquad (\text{for } -1 < v < 1) \\
\implies v^2 &= \frac{1}{3} + \frac{4}{\pi^2}(-\cos \pi v + \frac{1}{4}\cos 2\pi v - \frac{1}{9}\cos 3\pi v + \ldots) \qquad (\text{for } -1 < v < 1).
\end{aligned}$$

**11.2.17)** Funksjonen er like og har periode $P = 2L = 2$. Dermed er $b_n = 0$ for $n = 1, 2, 3, \ldots$
Sidan $f(x) = 1 - |x|$ for $x \in [-1, 1]$, er

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)dx \overset{\text{like}}{=} \int_0^1 f(x)dx = \int_0^1 (1-x)dx = \frac{1}{2}$$

$$
\begin{aligned}
a_n &= \frac{1}{1}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{1}dx \\
&\overset{\text{like}}{=} 2\int_0^1 f(x)\cos n\pi x dx \\
&= 2\int_0^1 (1-x)\cos n\pi x dx \\
&= 2\left[(1-x)\frac{1}{n\pi}\sin n\pi x\right]_0^1 - 2\int_0^1 (-1)\frac{1}{n\pi}\sin n\pi x dx \\
&= 0 + \frac{2}{n\pi}\left[\frac{-1}{n\pi}\cos n\pi x\right]_0^1 \\
&= \frac{2}{(n\pi)^2}(1-(-1)^n)
\end{aligned}
$$

Dvs.

$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x \\
&= \frac{1}{2} + \frac{4}{\pi^2}\left(\cos\pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5\pi x + ...\right)
\end{aligned}
$$

**11.2.25)** Vi startar med cosinusrekka:

$$
a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} \pi - x\ \mathrm{d}x = \frac{1}{\pi}\left[\pi x - \frac{1}{2}x^2\right]_0^{\pi} = \frac{1}{2}\pi
$$

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\ \mathrm{d}x \\
&= \frac{1}{\pi}\left[\frac{\pi}{n}\sin nx - \frac{nx\sin nx + \cos nx}{n^2}\right]_0^{\pi} = \frac{2(1+(-1)^{n+1})}{\pi n^2},\ n \geq 1 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ \mathrm{d}x = 0,\ n \geq 1 \\
\Rightarrow f(x) &= \frac{1}{2}\pi + \sum_{n=0}^{\infty} \frac{4}{\pi(2n+1)^2}\cos(2n+1)x
\end{aligned}
$$

Så reknar vi ut sinusrekka:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x \overset{\text{f odde}}{=} 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ \mathrm{d}x \overset{\text{f odde}}{=} 0$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} (\pi - x)\sin nx\ \mathrm{d}x$$

$$= \frac{2}{\pi}\Big[-\frac{\pi}{n}\cos nx + \frac{\sin nx + nx\cos nx}{n^2}\Big]_0^{\pi} = \frac{2}{n},\ n \geq 1$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty}\frac{2}{n}\sin nx$$

**13.1.9)**

$$z_1^2 = (-2+5i)^2 = (-2)^2 + 2(-2)5i + (5i)^2 = 4 - 25 - 20i$$

$$\mathrm{Re}(z_1^2) = -21$$

$$(\mathrm{Re}\, z_1)^2 = (-2)^2 = 4$$

**13.1.12)**

$$\frac{z_1}{z_2} = \frac{-2+5i}{3-i}$$

$$= \frac{(-2+5i)(3+i)}{(3-i)(3+i)}$$

$$= \frac{-6-2i+15i-5}{10}$$

$$= -\frac{11}{10} + \frac{13}{10}i$$

$$\frac{z_2}{z_1} = \frac{3-i}{-2+5i}$$

$$= \frac{(3-i)(2+5i)}{(-2+5i)(2+5i)}$$

$$= \frac{6+15i-2i+5}{-4-25}$$

$$= -\frac{11}{29} - \frac{13}{29}i$$

**13.1.18)**

$$(1+i)^{16}z^2 = (2i)^8 z^2$$

$$= 256(x^2 + 2xyi - y^2)$$

$$\Longrightarrow \mathrm{Re}((1+i)^{16}z^2) = 256(x^2 - y^2)$$

Alternativt kan ein skrive $i+1 = \sqrt{2}e^{i\pi/4}$ og bruke det til å rekne ut $(1+i)^{16}$.

**13.1.19)**

$$\begin{aligned}\frac{z}{\bar{z}} &= \frac{x+yi}{x-yi} \\ &= \frac{(x+yi)^2}{(x-yi)(x+yi)} \\ &= \frac{x^2-y^2+2xyi}{x^2+y^2}\end{aligned}$$

Vi får dermed

$$\mathrm{Re}\left(\frac{z}{\bar{z}}\right) = \frac{x^2-y^2}{x^2+y^2}$$

og

$$\mathrm{Im}\left(\frac{z}{\bar{z}}\right) = \frac{2xy}{x^2+y^2}.$$

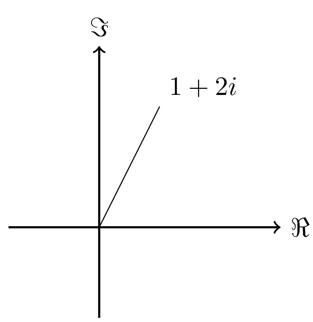
**13.2.3)** $2i = 2(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$, $-2i = 2(\cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2})$

**13.2.8)**

$$\frac{7+4i}{3-2i} = \frac{7+4i}{3-2i}\frac{3+2i}{3+2i} = 1+2i$$

$r = \sqrt{1^2+2^2} = \sqrt{5}$ og $\theta = \arctan(2)$ så vi får

$$\sqrt{5}(\cos(\arctan(2)) + i\sin(\arctan(2)))$$




**13.2.21)**

$$\begin{aligned}&|1-i| = \sqrt{1^2+1^2} = \sqrt{2}, \quad \arg(1-i) = \arctan\frac{-1}{1} = -\frac{\pi}{4} \\ \implies &1-i = \sqrt{2}e^{i\left(-\frac{\pi}{4}+n\cdot 2\pi\right)}, \quad n \in \mathbb{Z}\end{aligned}$$

La $w = \sqrt[3]{1-i} = Re^{i\phi}$

$$\begin{aligned}
w^3 &= 1 - i \\
\iff R^3 e^{i3\phi} &= \sqrt{2} e^{i\left(-\frac{\pi}{4} + n\cdot 2\pi\right)} \\
\implies R^3 &= \sqrt{2}, \quad 3\phi = -\frac{\pi}{4} + n \cdot 2\pi \\
\implies w &= 2^{\frac{1}{6}} e^{i\left(-\frac{\pi}{12} + n\cdot\frac{2}{3}\pi\right)}, \quad n \in \mathbb{Z}
\end{aligned}$$

Dermed finnes det tre ulike røtter. F.eks med $n = 0, 1, 2$:

$$\sqrt[3]{1-i} = \{2^{\frac{1}{6}} e^{-i\frac{\pi}{12}}, \quad 2^{\frac{1}{6}} e^{i\frac{7\pi}{12}}, \quad 2^{\frac{1}{6}} e^{i\frac{15\pi}{12}}\}$$

**D)** Merk at $f$ er ein like funksjon med periode $p = 2L = 4 \Rightarrow L = 2$. Fourierkoeffisientane er som følger:

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx = 2\frac{1}{2\cdot 2}\int_0^2 (1-x)dx = \underline{0}, \\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \int_0^2 (1-x)\cos\left(\frac{n\pi x}{2}\right)dx \\
&= \left[\frac{2}{n\pi}(1-x)\sin\left(\frac{n\pi x}{2}\right)\right]_0^2 - \frac{2}{n\pi}\int_0^2 (-1)\sin\left(\frac{n\pi x}{2}\right)dx \\
&= (\frac{2}{n\pi})^2\left[\cos\left(\frac{n\pi x}{2}\right)\right]_0^2 = (\frac{2}{n\pi})^2(1-\cos(n\pi)) \\
&= (\frac{2}{n\pi})^2(1-(-1)^n) = \underline{\begin{cases} \frac{8}{\pi^2}\frac{1}{(2m+1)^2}, & n = 2m+1 \\ 0, & n = 2m, \end{cases}} \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \underline{0}.
\end{aligned}$$

Fourierrekka til $f$ er dermed

$$\frac{8}{\pi^2}\sum_0^\infty \frac{\cos\left(\frac{2m+1}{2}\pi x\right)}{(2m+1)^2} = \frac{8}{\pi^2}\left(\cos\left(\frac{\pi x}{2}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{2}\right) + \dots\right).$$