Løysingsforslag, øving 1

September 1, 2016

6.1.1) $\mathscr{L}(2t+8) = 2\mathscr{L}(t) + 8\mathscr{L}(1) = \frac{2}{s^2} + \frac{8}{s}$. Detaljane i utrekningane er på side 205-207 i Kreyszig.

6.1.7) Formel for vinkelsum gir $\cos(\omega t + \theta) = \cos(\omega t)\cos(\theta) - \sin(\omega t)\sin(\theta)$. Merk at vi kan behandle $\cos(\theta)$ og $\sin(\theta)$ som konstantar under.

$$\begin{aligned}
\mathscr{L}(\cos(\omega t + \theta)) &= \mathscr{L}(\cos(\omega t)\cos(\theta) - \sin(\omega t)\sin(\theta)) \\
&= \cos(\theta)\mathscr{L}(\cos(\omega t)) - \sin(\theta)\mathscr{L}(\sin(\omega t)) \\
&= \cos(\theta)\frac{s}{s^2+\omega^2} - \sin(\theta)\frac{\omega}{s^2+\omega^2} \\
&= \frac{s\cos(\theta) - \omega\sin(\theta)}{s^2+\omega^2}
\end{aligned}$$

6.1.12) Vi bruker delvis integrasjon.

$$\begin{aligned}
\mathscr{L}(f) &= \int_0^1 e^{-st} t\,dt + \int_1^2 e^{-st}dt \\
&= -\frac{1}{s}e^{-st}t\Big|_0^1 - \int_0^1 \left(-\frac{1}{s}e^{-st}\right)dt - \frac{1}{s}e^{-st}\Big|_1^2 \\
&= -\frac{1}{s}e^{-s} + \frac{1}{s}\left(-\frac{1}{s}e^{-st}\Big|_0^1\right) - \frac{1}{s}e^{-2s} + \frac{1}{s}e^{-s} \\
&= \frac{1}{s^2}\left(1 - e^{-s}\right) - \frac{1}{s}e^{-2s}
\end{aligned}$$

6.1.21) Eit døme er $f(t) = e^{t^2}$. Sidan $e^{-st}e^{t^2}$ går mot ∞ når $t \to \infty$ for alle s, finst ikkje $\mathscr{L}(f)$.

6.1.22) Det ser ut til at dei i Table 6.1, formel 5 eigentleg meiner at $a+1$ skal vere positiv. Dermed kan vi bruke formel 5 direkte og få $\mathscr{L}(1/\sqrt{t}) = \Gamma(1/2)/\sqrt{s} = \sqrt{\pi}/\sqrt{s}$. $1/\sqrt{t}$ tilfredsstiller ikkje (2) på side 209 nær 0, men har likevel veldefinert Laplacetransformert, så krava i Theorem 3 er ikkje nødvendige for at den Laplacetransformerte skal eksistere.

6.1.23) Plugger $f(ct)$ inn i (1) side 204:

$$\mathscr{L}(f(ct)) = \int_0^\infty e^{-st} f(ct) dt.$$

Variabelskifte til $\tau = ct$ gir

$$\int_0^\infty e^{-(s/c)\tau} f(\tau) \frac{d\tau}{c} = \frac{1}{c} F\left(\frac{s}{c}\right).$$

Vi har

$$\mathscr{L}(\cos(t)) = \frac{s}{s^2+1}.$$

Formelen vi viste over gir

$$\begin{aligned} \mathscr{L}(\cos(\omega t)) &= \frac{1}{\omega} \frac{s/\omega}{(s/\omega)^2+1} \\ &= \frac{s}{s^2+\omega^2}. \end{aligned}$$

6.1.25)

$$\begin{aligned} \frac{0.2s+1.4}{s^2+1.96} &= 0.2\frac{s}{s^2+1.96} + \frac{1.4}{s^2+1.96} \\ &= 0.2\mathscr{L}(\cos(1.4t)) + \mathscr{L}(\sin(1.4t)) \\ &= \mathscr{L}(0.2\cos(1.4t) + \sin(1.4t)) \\ \Rightarrow f(t) &= 0.2\cos(1.4t) + \sin(1.4t) \end{aligned}$$

6.1.26) Delbrøkoppspaltar og bruker (6) i tabellen side 207.

$$\begin{aligned} \frac{5s+1}{s^2-25} &= \frac{12}{5}\frac{1}{s+5} + \frac{13}{5}\frac{1}{s-5} \\ &= \frac{12}{5}\mathscr{L}(e^{-5t}) + \frac{13}{5}\mathscr{L}(e^{5t}) \\ \Rightarrow f(t) &= \frac{12}{5}e^{-5t} + \frac{13}{5}e^{5t} \end{aligned}$$

Ein kan også gjer det følgjande, med ekvivalent svar:

$$\begin{aligned}
\frac{5s+1}{s^2-25} &= 5\frac{s}{s^2-25} + \frac{1}{5}\frac{5}{s^2-25} \\
&= 5\mathscr{L}(\cosh(5t)) + \frac{1}{5}\mathscr{L}(\sinh(5t)) \\
&= \mathscr{L}(5\,\cosh(5t) + \frac{1}{5}\,\sinh(5t)) \\
\Rightarrow f(t) &= 5\,\cosh(5t) + \frac{1}{5}\,\sinh(5t)
\end{aligned}$$

6.1.41) Vi har

$$\begin{aligned}
\frac{\pi}{s^2+4s\pi+3\pi^2} &= \frac{\pi}{(s+2\pi)^2-\pi^2} \\
&= F(s-(-2\pi)),
\end{aligned}$$

der $F(s) = \pi/(s^2-\pi^2) = \mathscr{L}(\sinh(\pi t))$. Theorem 2 side 208 gir

$$\mathscr{L}^{-1}\left(\frac{\pi}{s^2+4s\pi+3\pi^2}\right) = e^{-2\pi t}\,\sinh(\pi t)$$

Alternativt:

$$\begin{aligned}
\frac{\pi}{s^2+4s\pi+3\pi^2} &= \frac{\pi}{(s+2\pi)^2-\pi^2} \\
&= \frac{1}{2}\left(\frac{1}{(s+2\pi)-\pi} - \frac{1}{(s+2\pi)+\pi}\right) \\
&= \frac{1}{2}\left(\frac{1}{s+\pi} - \frac{1}{s+3\pi}\right) \\
&= \frac{1}{2}\left(\mathscr{L}(e^{-\pi t}) - \mathscr{L}(e^{-3\pi t})\right) \\
\Rightarrow \mathscr{L}^{-1}\left(\frac{\pi}{s^2+4s\pi+3\pi^2}\right) &= \frac{1}{2}\left(e^{-\pi t} - e^{-3\pi t}\right)
\end{aligned}$$

6.2.9) Laplacetransformen av høgre side:

$$\mathscr{L}(4t-8) = \frac{4}{s^2} - \frac{8}{s}$$

La $Y(s) = \mathscr{L}(y(t))$. Laplacetransformen av venstre side:

$$\begin{aligned}
\mathscr{L}(y'' - 3y' + 2y) &= s^2Y - sy(0) - y'(0) - 3(sY - y(0)) + 2Y \\
&= s^2Y - 2s - 7 - 3(sY - 2) + 2Y \\
&= (s^2 - 3s + 2)Y - 2s - 1
\end{aligned}$$

Desse uttrykka skal vere like, så vi får

$$
\begin{aligned}
Y(s) &= \left(2s + 1 + \frac{4}{s^2} - \frac{8}{s}\right) \frac{1}{s^2 - 3s + 2} \\
&= \frac{s^2(2(s-3) + 7) + 4 - 8s}{s^2(s^2 - 3s + 2)} \\
&= \frac{2s^3 + s^2 - 8s + 4}{s^2(s-1)(s-2)} \\
&= \frac{-s+2}{s^2} + \frac{1}{s-1} + \frac{2}{s-2} \\
&= \frac{-1}{s} + \frac{2}{s^2} + \frac{1}{s-1} + \frac{2}{s-2} \\
\Rightarrow y(t) &= -1 + 2t + e^t + 2e^{2t}
\end{aligned}
$$

Bruker (7) på side 214. Vi har

$$
\begin{aligned}
Q(s) &= \frac{1}{s^2 - 3s + 2} \\
R(s) &= \mathscr{L}(4t - 8) = \frac{4}{s^2} - \frac{8}{s} \\
\Rightarrow Y(s) &= \left[2(s-3) + 7 + \frac{4}{s^2} - \frac{8}{s}\right] \frac{1}{s^2 - 3s + 2} \\
&= \frac{s^2(2(s-3) + 7) + 4 - 8s}{s^2(s^2 - 3s + 2)} \\
&= \frac{2s^3 + s^2 - 8s + 4}{s^2(s-1)(s-2)} \\
&= \frac{-s+2}{s^2} + \frac{1}{s-1} + \frac{2}{s-2} \\
&= \frac{-1}{s} + \frac{2}{s^2} + \frac{1}{s-1} + \frac{2}{s-2} \\
\Rightarrow y(t) &= -1 + 2t + e^t + 2e^{2t}
\end{aligned}
$$

6.2.27) Vi har

$$
\frac{s+8}{s^4 + 4s^2} = \frac{1}{s^2} F(s),
$$

der $F(s) = \frac{s+8}{s^2+4}$.

$$
\begin{aligned}
\mathscr{L}^{-1}(F) &= \cos(2t) + 4\sin(2t) \\
\mathscr{L}^{-1}\left(\frac{1}{s}F\right) &= \int_0^t (\cos(2\tau) + 4\sin(2\tau)) \\
&= \frac{1}{2}\sin(2\tau) - 2\cos(2\tau)\Big|_0^t \\
&= \frac{1}{2}\sin(2t) - 2\cos(2t) + 2 \\
\mathscr{L}^{-1}\left(\frac{1}{s^2}F\right) &= \int_0^t \left(\frac{1}{2}\sin(2\tau) - 2\cos(2\tau) + 2\right) dt \\
&= -\frac{1}{4}\cos(2\tau) - \sin(2\tau) + 2\tau\Big|_0^t \\
&= -\frac{1}{4}\cos(2t) - \sin(2t) + 2t + \frac{1}{4}.
\end{aligned}
$$

6.3.5) Kall funksjonen f. Då er $f(t) = e^{-t}(u(t) - u(t-\pi))$, og

$$
\begin{aligned}
\mathscr{L}(f) &= \int_0^\infty e^{-st}e^{-t}(u(t) - u(t-\pi))dt \\
&= \int_0^\pi e^{-st}e^{-t}dt \\
&= -\frac{1}{s+1}e^{-(s+1)t}\Big|_0^\pi \\
&= \frac{1}{s+1}\left(1 - e^{-(s+1)\pi}\right)
\end{aligned}
$$

6.3.13) Bruker Theorem 1 på side 219.

$$
\begin{aligned}
\frac{4(1-e^{-\pi s})}{s^2+4} &= 2\frac{2}{s^2+4} - 2e^{-\pi s}\frac{2}{s^2+4} \\
&= 2\mathscr{L}(\sin(2t)) - 2e^{-\pi s}\mathscr{L}(\sin(2t)) \\
&= 2\mathscr{L}(\sin(2t)) - 2\mathscr{L}(\sin(2(t-\pi))u(t-\pi)) \\
\Rightarrow f(t) &= 2\sin(2t) - 2\sin(2(t-\pi))u(t-\pi) \\
&= 2\sin(2t)(1 - u(t-\pi))
\end{aligned}
$$

6.3.40) Følger Example 1 og bruker (1') side 94. Vi får likninga

$$
i' + 2i + 10\int_0^t i(\tau)d\tau = 255\sin(t)(u(t) - u(t-2\pi)).
$$

La $I(s) = \mathscr{L}(i)$. Vi får

$$
\begin{aligned}
sI + 2I + \frac{10}{s} I &= 255 \left(\frac{1}{s^2+1} + e^{-2\pi s} \frac{1}{s^2+1} \right) \\
\left(s + 2 + \frac{10}{s} \right) I &= \frac{255(1 - e^{-2\pi s})}{s^2+1} \\
I(s) &= 255(1 - e^{-2\pi s}) \frac{s}{(s^2+2s+10)(s^2+1)} \\
&= 255(1 - e^{-2\pi s}) \left(\frac{-9s/85 + -20/85}{s^2+2s+10} + \frac{9s/85 + 2/85}{s^2+1} \right) \\
&= 3(1 - e^{-2\pi s}) \left(\frac{-9s + -20}{s^2+2s+10} + \frac{9s+2}{s^2+1} \right) \\
&= (1 - e^{-2\pi s}) \left(\frac{-27(s+1) + -33}{(s+1)^2+9} + \frac{27s+6}{s^2+1} \right) \\
&= (1 - e^{-2\pi s}) \left(-27\mathscr{L}(e^{-t}\cos(3t)) - 11\mathscr{L}(e^{-t}\sin(3t)) + 27\mathscr{L}(\cos(t)) + 6\mathscr{L}(\sin(t)) \right) \\
&= (1 - e^{-2\pi s}) \mathscr{L} \left(-27e^{-t}\cos(3t) - 11e^{-t}\sin(3t) + 27\cos(t) + 6\sin(t) \right) \\
\Rightarrow i(t) &= -27e^{-t}\cos(3t) - 11e^{-t}\sin(3t) + u(t-2\pi) \left(27e^{-t+2\pi}\cos(3t) + 11e^{-t+2\pi}\sin(3t) \right) \\
&\quad + (1 - u(t-2\pi)) \left(27\cos(t) + 6\sin(t) \right).
\end{aligned}
$$

Her har vi brukt delbrøkoppspalting, formlane side 207, Theorem 1 side 206 (linearitet), og til slutt Theorem 1 side 219, der vi utnyttar at $\sin(n(t+2\pi)) = \sin(nt)$ for heiltal n.

A) Tek Laplacetransformen av høgre side:

$$
\begin{aligned}
R(s) = \mathscr{L}(r(t)) &= \mathscr{L}(u(t) + u(t-1) - 2u(t-2)) \\
&= \left(1 + e^{-s} - 2e^{-2s} \right) \frac{1}{s}.
\end{aligned}
$$

La $Y(s) = \mathscr{L}(y(t))$. Tek Laplacetransformen av venstre side:

$$
\begin{aligned}
\mathscr{L}(y'' + y' - 2y) &= s^2 Y - sy(0) - y'(0) + sY - y(0) - 2Y \\
&= (s^2 + s - 2)Y - s - 2.
\end{aligned}
$$

Set venstre side lik høgre side og får

$$
\begin{aligned}
Y(s) &= \frac{s + 2 + \left(1 + e^{-s} - 2e^{-2s} \right)/s}{s^2 + s - 2} \\
&= \frac{1}{s-1} + \frac{1 + e^{-s} - 2e^{-2s}}{s(s^2+s-2)}.
\end{aligned}
$$