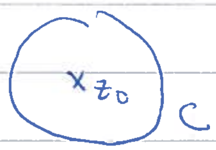
Lecture 25 16.3 Residue.

- Reminder: z_0 - isolated singularity of f

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-z_0)^n$$



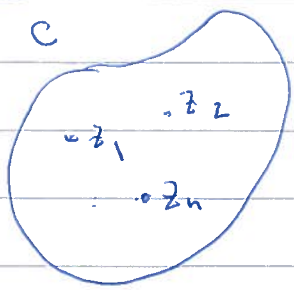
$$\operatorname{Res}_{z_0} f = \frac{1}{2i\pi} \int_C f(z)\,dz = a_{-1}$$

- z_0-simple pole i.e. $f(z) = \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \dots$

$$\Rightarrow \operatorname{Res}_{z_0} f = \lim_{z \to z_0} f(z)(z-z_0)$$

- In particular $f(z) = \dfrac{h(z)}{g(z)}$, $h(z_0) \neq 0$, $g(z_0) = 0$, $g'(z_0) \neq 0$

$$\Rightarrow \operatorname{Res}_{z_0} f = \frac{h(z_0)}{g'(z_0)}$$



$f(z)$ is analytic inside C except isolated singularities at $z_1, z_2, \dots, z_n$ $\Rightarrow$

$$\frac{1}{2i\pi} \int_C f(z)\,dz = \sum_{j=1}^{n} \operatorname{Res}_{z_j} f$$

New:

Pole of order N:

$$f(z) = \frac{a_{-N}}{(z-z_0)^N} + \frac{a_{-N+1}}{(z-z_0)^{N-1}} + \dots + a_0 + a_1(z-z_0) + \dots$$

$$a_{-N} \neq 0.$$

Rule for calculating residues:

$$\operatorname{Res}_{z_0} f(z) = \frac{1}{(N-1)!} \lim_{z \to z_0} \frac{d^{N-1}}{dz^{N-1}} \left((z-z_0)^N f(z) \right)$$

Not very elegant :(

~~Example / X f(z)~~

Order of zero and order of pole

$$f(z) = \frac{h(z)}{g(z)},$$ $h(z_0) \neq 0$, $g(z_0) = 0$ and has zero of order N at z_0

i.e. $g(z) = (z-z_0)^N \left[c_N + c_{N+1}(z-z_0) + \dots \right]$

$\Longleftrightarrow$ $f(z)$ has pole of order N

i.e. $\boxed{\text{order of pole of } 1/g \text{ is order of zero of } g}$

Example:

$$f(z) = \frac{2+z}{z^3+z}, \quad z_0 = 0$$

$$f(z) = \frac{2+z}{z^2(z+1)} = \frac{h(z)}{g(z)} \qquad h(z) = \frac{2+z}{z+1}$$

$h(0) \neq 0$

$g(z) = z^2$ – zero of order 2

$\Rightarrow f$ has pole of order 2.

Calculating residues:

$$\operatorname{Res}_0 f(z) = \lim_{z \to 0} \frac{d}{dz}\left(z^2 f(z)\right)$$

$$z^2 f(z) = \frac{2+z}{z+1} = \cancel{\frac{2+z}{z+1}}\ 1 + \frac{1}{z+1}$$

$$\left.\begin{aligned} &\left(1 + \frac{1}{z+1}\right)' = -\frac{1}{(z+1)^2} \\ &-\frac{1}{(z+1)^2}\Big|_{z=0} = -1 \end{aligned}\right\} \Rightarrow \operatorname{Res}_0 f = -1$$

General instruction

Step 1 Define the order N of the pole

Step 2 Apply formula with the proper N.

Surprise!

One can evaluate real integrals by using methods of complex analysis.

Example: $I = \int_0^{2\pi} \frac{dt}{2+\cos t}$

Euler: $\cos t = \frac{1}{2}(e^{it} + e^{-it})$

Idea: Change of variable: $z = e^{it}$

t - runs $[0, 2\pi] \Rightarrow z = e^{it}$ - runs over the unit circle C

Denominator:

$$2 + \cos t \longrightarrow 2 + \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left(4 + z + \frac{1}{z}\right)$$

Numerator:

$$z = e^{it} \Rightarrow dz = i\underbrace{e^{it}}_{z}\,dt = iz\,dt \Rightarrow$$

$$dt = \frac{1}{iz}\,dz$$

Integration path

$$[0, 2\pi] \longrightarrow C = \{z : |z| = 1\}$$

Finally

$$I = \frac{2}{i}\int_C \frac{dz}{z\left(4+z+\frac{1}{z}\right)} =$$

$$= \frac{2}{i}\int_C \frac{dz}{z^2+4z+1}$$

|| We obtained an integral of an analytic function over a closed curve!

Finding singular points:

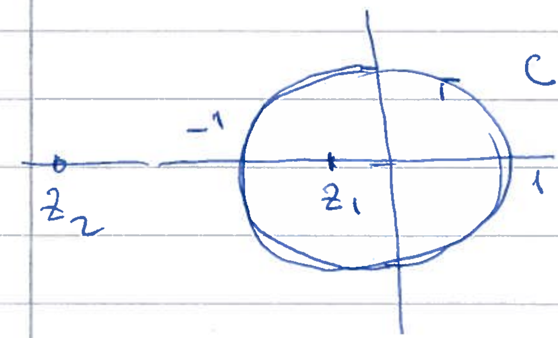
$z^2+4z+1=0$ - solutions of this are singular points.

$$z^2+4z+4-3=0 \Rightarrow (z+2)^2=3 \Rightarrow z=-2\pm\sqrt{3}$$

$$z_1=-2+\sqrt{3} \; ; \quad z_2=-2-\sqrt{3}$$

Making picture

$(\sqrt{3} \sim 1{,}7)$



$C=\{z : |z|=1\}$.

z_1 - inside C

z_2 - outside C

$$\Rightarrow \cancel{I = \frac{2}{i}\int_C z}$$

$$I = \frac{2}{i}\int_C \frac{dz}{z^2+4z+1} = 4\pi \frac{1}{2i\pi}\int_C \frac{dz}{z^2+4z+1} =$$

$$= 4\pi \operatorname{Res}_{z_1} \frac{1}{z^2+4z+1}$$

Calculating residues

z_1 - simple pole

~~Denom~~ Denominator $g(z) = z^2 + 4z + 1$

$$g'(z) = 2z + 4$$

$$g'(z_1) = 2(-2+\sqrt{3}) + 4 = 2\sqrt{3}$$

$$\operatorname{Res}_{z_1} \frac{1}{z^2+4z+4} = \frac{1}{2\sqrt{3}}$$

$$I = \frac{2\pi}{\sqrt{3}}$$

General pattern

$$I = \int_0^{2\pi} F(\cos t, \sin t)\, dt$$

where F is a rational function of $\cos t$ and $\sin t$

i.e. ratio of two polynomials of $\cos t$ and $\sin t$

Step 1 Change of variables

$$e^{it} = z \implies \cos t \to \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin t \to \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$dt \to \frac{1}{iz}\, dz$$

$$[0, 2\pi] \to C = \{z : |z| = 1\}$$

$$F(\cos t, \sin t)\, dt \to R(z)\, dz$$

where $R(z)$ some rational function of z

$$I = \int_C R(z)\, dz$$

Step 2

Find poles of $R(z)$: $z_1, \dots, z_n$

and determine which of them are

inside of the unit disk: $z_1, \dots, z_p$, say.

then

$$I = 2i\pi \sum_{j=1}^{p} \operatorname{Res}_{z_j} R(z)$$

Step 3

Calculate the residues and find I.

Remark: Rational function of $\cos t$ and $\sin t$ may be hidden. If you have $\cos 2t$ for example, replace it by polynomial of $\cos t$.

Improper integrals

Example:

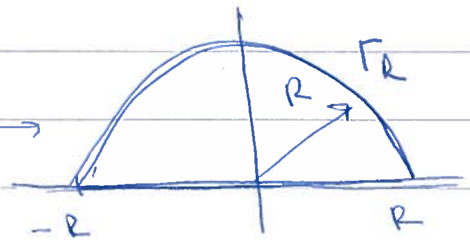
$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$

Disaster: We do NOT have a closed curve to apply the Cauchy integral theorem!

Suggestion: Make a closed curve!

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^4}$$

Take C_R $\longrightarrow$



$C_R = [-R, R] \cup \Gamma_R$ - closed curve!

$\Gamma_R = \{z : |z| = R, \ \mathrm{Im}\, z > 0\}$

In addition

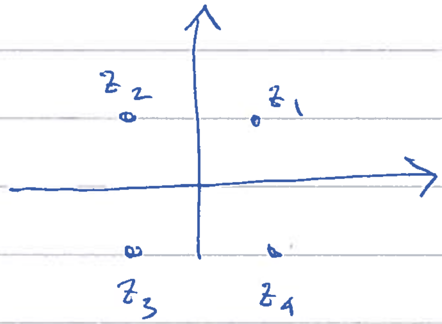
$$\int_{\Gamma_R} \frac{dz}{1+z^4} \longrightarrow 0 \quad \text{as } R \to \infty$$

Follows from the ML theorem!

Singular points:

$$1+z^4=0 \Rightarrow z_1 = e^{i\frac{\pi}{4}}, \quad z_2 = e^{i\frac{3\pi}{4}}, \quad z_3 = e^{i\frac{5\pi}{4}},$$

$$z_4 = e^{i\frac{7\pi}{4}}$$



z_1 and z_2 are inside C_R

$$\int_{C_R} \frac{dz}{1+z^2} = 2i\pi\left(\operatorname{Res}_{z_1} \frac{1}{1+z^4} + \operatorname{Res}_{z_2} \frac{1}{1+z^4}\right)$$

$$= 2i\pi\left(\frac{1}{4(e^{i\pi/4})^3} + \frac{1}{4(e^{i3\pi/4})^3}\right) = \frac{\pi}{\sqrt{2}}$$

Now

$$\int_{C_R} = \int_{-R}^{R} + \int_{\Gamma_R}$$

We have

$$\int_{-R}^{R} \frac{dx}{1+x^2} = -\int_{\Gamma_R} \frac{dz}{1+z^4} = \frac{\pi}{\sqrt{2}}$$

$$\downarrow \qquad\qquad \downarrow R\to\infty \qquad\qquad \underline{R\to\infty}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} \qquad\qquad 0$$

And finally

$$\cancel{\int_{-R}^{R}} \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{\sqrt{2}}$$

* * *

General condition for $\int_{\Gamma_R} f(z)\,dz \to 0$ as $R\to\infty$.

$$\left(\max_{|z|=R} |f(z)|\right) \cdot R \to 0$$

For example: $|f(z)| \sim \frac{1}{|z|^2}$ as $|z|\to\infty$.

<u>Example:</u> Rational function:

$$f(z) = \frac{P(z)}{Q(z)}, \quad \S$$

P, Q - polynomials, $\deg Q \geq \deg P + 2$