Lecture 22

1. Power series - summary.

- $$S(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n$$

- Domain of convergence: disk around z_0
  $\{z : |z - z_0| < R\}$.
  ↑ radius of convergence.

- Formula for R: $R = \lim_{n \to \infty} \frac{|a_n|}{|a_{n+1}|}$ if exists.

- The sum $S(z)$ is analytic in $\{z : |z - z_0| < R\}$.

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- $f(z)$ analytic in $\{z : |z - z_0| < R\}$ ⟹

$$\Rightarrow f(z) = \sum_{0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n$$ Taylor series

- Taylor series at z_0 converges in the biggest disk centered at z_0 where f is analytic

- Uniqueness: $f^{(n)}(z_0) = 0$, $n = 0, 1, 2, \ldots \Rightarrow f(z) \equiv 0$.

Example:

$$\sin z = z - \frac{1}{6} z^3 + \frac{1}{120} z^5 - \dots =$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}$$

$$\frac{\sin z}{z} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}$$ – this is an analytic function!

Sometimes we can divide by function which vanishes at z_0 !

Zeros of analytic functions (from 16.2).

1. $f(z)$ - analytic in $\{z : |z - z_0| < R\} \Rightarrow$

[illegible]

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z-z_0)^2}{2} f''(z_0) + \dots$$

Def: z_0 is zero of f if $f(z_0) = 0$

~~Related part of~~ 16.2

$$f(z_0)=0 \Rightarrow f(z) = \frac{1}{k!} f^{(k)}(z_0)(z-z_0)^k +$$

$$+ \frac{1}{(k+1)!} f^{(k+1)}(z_0)(z-z_0)^{k+1} + \dots$$

k is the number of first non-zero derivative at z_0.

k is <u>order of zero</u> at z_0

<u>Examples:</u> $\cos z - 1$ has zero of order 2 at 0.

~~[illegible]~~

<u>Fact:</u> Zeros are isolated: if $f(z_0)=0$ then for some ε $f(z)\neq 0$ for $|z-z_0|<\varepsilon$

<u>Why?</u> $$f(z) = \sum_{n=k}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z-z_0)^n =$$

$$= (z-z_0)^k \sum_{n=k}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z-z_0)^{n-k} =$$

$$= \underbrace{(z-z_0)^k}_{\text{does not vanish if except of at } z_0} \underbrace{\Big[\underbrace{\frac{1}{n!} f^{(n)}(z_0)}_{\neq 0} + \underbrace{\frac{1}{(n+1)!} f^{(n+1)}(z_0)(z-z_0) + \dots}_{\text{small as } |z-z_0|<\varepsilon} \Big]}_{\text{does not vanish near } z_0}$$

Division of analytic functions:

$$\left.\begin{array}{l} f(z), g(z) \text{ - analytic near } z_0 \\ f(z_0)=0,\ g(z_0)=0 \\ f \text{ has zero of order } k \\ g \text{ has zero of order } l \end{array}\right\} k > l$$

$\Rightarrow \dfrac{f}{g}$ - analytic function near z_0.

Some techniques

Examples / problems.

1. $\dfrac{1}{1-z^2}$ - find power series at 0

2. $f(z) = \dfrac{1}{\sin z}$ find convergence radius for the Taylor series centered at $z_0 = (1+i)$

3. One more example

$$\cos\sqrt{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n)!}$$ - this is an analytic function.

4. Prove: $\dfrac{1}{(1-z)^2} = \displaystyle\sum_{n=0}^{\infty} (n+1) z^n \,, \quad |z| < 1$

5. Find Taylor series ~~for~~ centered at 0 for $f(z) = \sin^2 z$

We start Ch. 16 Laurent series.

Motivation: $\frac{1}{z} = z^{-1}$ – this is also a "power" series.

• ~~Reminder~~: ~~Representation~~ Just negative powers

$\boxed{\text{Can this be included into theory.?}}$

Reminder Representation of function which is analytic in $\{z : r < |z - z_0| < R\} < \infty$

$$f(z) = \underbrace{\frac{1}{2\pi i} \oint_{|\zeta - z_0| = R} \frac{f(\zeta)}{\zeta - z} d\zeta}_{f^+(z)} - \underbrace{\frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z} d\zeta}_{f^-(z)}$$

$f_+(z)$ is analytic in $|z| < R$

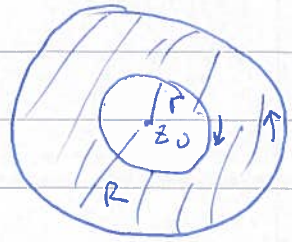
$f_-(z)$ is analytic in $|z| > r$

How can one develop power series for this case?

But we cannot speak about derivative at z_0!

Laurent series :

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$



$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

The same formula as for the Taylor series but :

- n runs from $-\infty$ to $+\infty$.

- C now consists of two circles :

  outer – counterclockwise

  inner – clockwise.