Forced oscillations

$y'' + cy' + ky = r(t)$, $\quad r(t) = r(t+2\pi)$

Idea: Expand $r(t)$ into Fourier series

$$r(t) = \sum_{-\infty}^{\infty} c_n e^{int}$$

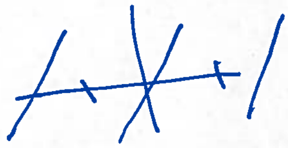
and solve each eq-n $y'' + cy' + ky = c_n e^{int}$

separately.

1. What kind of solution we can expect?

We need $c_0 = 0$

2. Example: $r(t) = t$, $-\pi < t < \pi$



Coefficients

$c_0 = 0$

$$n \neq 0 \Rightarrow c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} t e^{-int} dt =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} t \left(\frac{e^{-int}}{-in} \right)' dt =$$

$$= \frac{-1}{2i\pi n} t e^{-int} \Big|_{t=-\pi}^{\pi} + \frac{1}{2i\pi n} \underbrace{\int_{-\pi}^{\pi} e^{-int} dt}_{=0} =$$

$$= \frac{-1}{2i\pi n} \left(\pi (-1)^n + \pi (-1)^n \right) = \frac{i}{n} (-1)^n = (-1)^n \frac{i}{n}$$

$$r(t) = i \sum_{n \neq 0} \frac{(-1)^n}{n} e^{int}$$

Equation for each n

$$y'' + cy' + ky = c_n e^{int}$$

Periodic solution:

$$y_n = \alpha_n e^{int} \Bigg\} \Rightarrow$$

$$\Rightarrow \alpha_n = \frac{c_n}{k - n^2 + ic}, \quad y_n(t) = \frac{c_n}{k - n^2 + ic} e^{int}$$

Total solution:

$$\cancel{\boxed{y(t) = \sum_{n \neq 0} \frac{(-1)^n}{k}}} \qquad y(t) = \sum_{n \neq 0} \frac{c_n}{k - n^2 + ic} e^{+int}$$

In particular, for our case

$$y(t) = \sum_{n \neq 0} \frac{(-1)^n}{n} \frac{1}{k - n^2 + ic} e^{int}$$

Remark: If $c \ll 1$ and for some n_0 $|k - n_0^2| \ll 1$, i.e. k is very close to n_0^2, then contribution of the n_0th coefficient will be HUGE!

Exercise: Do the same problem with trig. Fourier series, then you have to look at y_n in the form

$$y_n(t) = \alpha_n \cos nt + \beta_n \sin nt.$$

Complex Fourier series for 2L-periodic functions

Nothing essentially new:

$$f(t+2L) = f(t), \quad -\infty < t < \infty$$

(Equivalently given $f(t)$ for $-\pi < t < \pi$, say).

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{\pi}{L}nt}$$

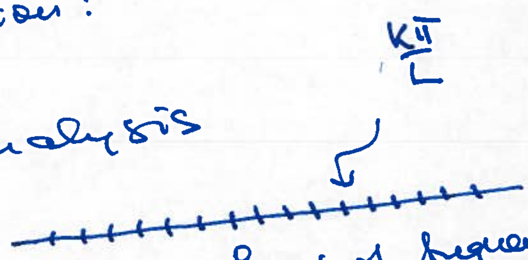
$$c_n := \frac{1}{2L}\int_{-L}^{L} f(t)\, e^{-i\frac{\pi}{L}nt}\, dt$$

$$S_N(f) = \sum_{-N}^{N} c_n e^{+i\frac{\pi}{L}nt} \to f, \quad N \to \infty$$

Fourier series: $f(t) = \sum_{-\infty}^{\infty} c_n e^{i\frac{\pi}{L}nt}$

Signal-analysis interpretation:

$f \rightsquigarrow \{c_n\}$ — analysis



set of frequences. $\{\frac{n\pi}{L}\}_n$

$\{c_n\} \rightsquigarrow f$ — synthes.

~~Digital Fourier transform~~

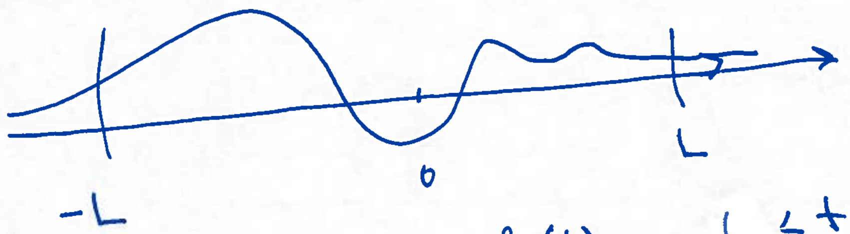
Comments about digitalization.

Fourier integral (Complex form, sec. 11.9)

$\boxed{\text{What happens if } f \text{ is NOT a periodic}}$? !

Assumption $f(t) \to 0$, $t \to \pm\infty$
"sufficiently fast"

Idea:
- Truncate f on $[-L, L]$
- Take Fourier series
- Look at what happens as $L \to \infty$.



$$f_L(t) = \begin{cases} f(t), & -L \le t < L \\ 2L \text{ periodic prolongation.} \end{cases}$$

$$f_L(t) = \sum_{n=-\infty}^{\infty} c_n(L)\, e^{i\frac{\pi}{L}nt}$$

$$c_n(L) = \frac{1}{2L}\underbrace{\int_{-L}^{L} f(t)\, e^{-i\frac{\pi}{L}nt}\, dt}$$

here I can ~~some~~ pass to the limit as $L \to \infty$

FOURIER TRANSFORM

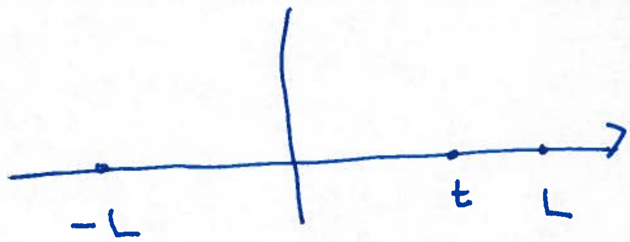
Denote:

$$(\mathcal{F}f)(w) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt$$

Then

$$c_n(L) \sim \frac{\sqrt{2\pi}}{2L} \hat{f}\left(\frac{\pi}{L} n\right)$$

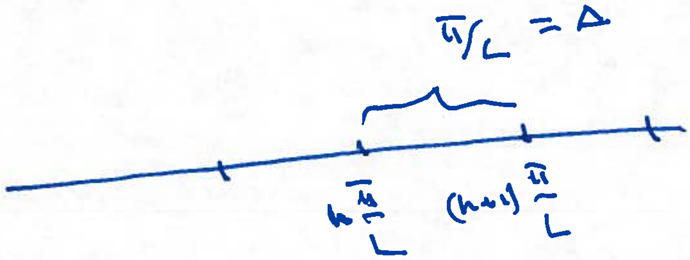
Reconstruction formula:



Let $|t| < L$. Then

$$f(t) = f_L(t) = \sum_{n=-\infty}^{\infty} c_n(L)\, e^{+i\frac{\pi}{L} n t} \sim$$

$$\sim \sum_{-\infty}^{\infty} \frac{\sqrt{2\pi}}{2L} \hat{f}\left(\frac{\pi n}{L}\right) e^{i\frac{\pi}{L} n t} =$$



$$= \underbrace{\sum_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \frac{\pi}{L} \hat{f}\left(\frac{\pi n}{L}\right) e^{+i\frac{\pi}{L} n t}}$$

This is just the integral sum

$$\xrightarrow{L \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwt}\, dw$$

Inverse Fourier transform

Basic formulas

- **Analysis:** Fourier transform

$$\mathcal{F}f(w) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt$$

- **Synthesis:** Inverse Fourier transform:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwt}\, dw .$$

⚠ **Danger:** Various normalizations for Fourier transform.

Comment: Inverse F.t. is just the same with + sign in exponent.

Energy preservation (Parseval equality for F.t.)

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |\hat{f}(w)|^2\, dw$$

Exercise: Obtain this from the Parseval relation for the Fourier series.

Basic example (Ideal low pass filter).

$$f(t) = \begin{cases} 1 & -T < t < T \\ 0, & |t| > T \end{cases}$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\omega} \, dt =$$

$$= \frac{-1}{\sqrt{2\pi}\, i\omega} e^{-it\omega} \Big|_{t=-T}^{T} =$$

$$= \frac{-1}{\sqrt{2\pi}\, i\omega} \left(e^{-iT\omega} - e^{iT\omega}\right) = \sqrt{\frac{2}{\pi}} \frac{e^{iT\omega} - e^{-iT\omega}}{2i\omega} =$$

$$= \underbrace{\sqrt{\frac{2}{\pi}} \frac{\sin T\omega}{\omega}}$$

This is ideal low pass filter.

We can write

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin T\omega}{\omega} e^{it\omega} \, d\omega = \begin{cases} 1, & |t| < T \\ 0 & \text{otherwise.} \end{cases}$$

Now this follows from the inverse F.t.

Later we will be able to ~~prove this~~ see this directly.

Two exam problems (#6 2009, #6 2010)

Find Fourier transforms of

- $$f(x) = \begin{cases} \sin 3x & -\pi < x < \pi \\ 0 & \text{otherwise} \end{cases}$$

- $$f(t) = e^{-|t|}$$

Convolution:

$f(t)$, $g(t)$ defined for $-\infty < t < \infty$.

$$h(t) = (f*g)(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

- If $f(t)=0$, $g(t)=0$ for $t<0$ this is the same as convolution for Laplace!

Fourier transform of convolution

$$\mathcal{F}(f*g)(\omega) = (\mathcal{F}f)(\omega)\cdot(\mathcal{F}g)(\omega).$$

Why function $\dfrac{\sin \Omega t}{t}$ is a low-pass filter

Derivative:

$$\mathcal{F}(f'(t))(\omega) = i\omega(\mathcal{F}f)(\omega).$$