6.4 Short impulses. Dirac δ-function. 6.5 Convolution

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Physical meaning of 2nd order differential equation

$$my''(t) + ky'(t) + \gamma y(t) = f(t), \;\; y(0) = y_0, y'(0) = y_1.$$

m - mass;

k - resistance (friction for example);

γ - elasticity;

f - external force;

y_0, y_1 - initial position and velocity.

Impulse: $\int_0^\infty f(t)dt$

Impulse - "measure of efforts made by the external force"

Modeling instant hit (hummer blow):

Hummer blow = instant hit with given impulse (for example impulse equal 1)

Idea: Take a very short hit of impulse 1, let its length approach zero and see what is going to happen
We model an instant hit at the moment t_0.Take

$$f_\Delta(t) = \begin{cases} 0, & t < t_0; \\ \frac{1}{\Delta}, & t_0 \le t < t_0 + \Delta; \\ 0, & t_0 + \Delta < t. \end{cases}$$

and then pass to the limit as $\Delta \to 0$.

Bad news and good news

Unfortunately

$$\lim_{\Delta \to 0} f_\Delta(t) = \begin{cases} \infty, & t = t_0, \\ 0, & t \neq t_0 \end{cases}$$

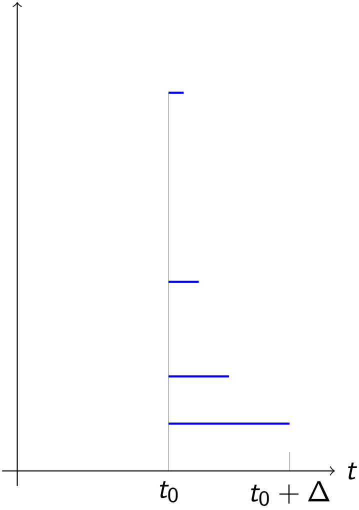
does not make much sense.

Fortunately

$\lim_{\Delta \to 0} (\mathcal{L} f_\Delta)(s)$ perfectly exists!

Hence we can use it in the right-hand side when solving ODE by using Laplace transform!

Dirac delta function



We consider the limit of those functions f_Δ as $\Delta \to 0$. This limit is called the Dirac delta function,

$$\delta_{t_0}(t) = \delta(t - t_0)$$

which is zero unless $t = t_0$ and satisfies

$$\int_{-\infty}^{\infty} \delta_{t_0}(t) dt = 1.$$

Laplace transform of the delta function

We can compute integrals of $\delta(t-t_0)g(t)$

$$\int_{-\infty}^{\infty}\delta(t-t_0)g(t)dt=\lim_{\Delta\to 0}\frac{1}{\Delta}\int_{t_0}^{t_0+\Delta}g(t)dt=g(t_0),$$

where g is a continuous function.
In particular, if $t_0>0$ then

$$\mathcal{L}\{\delta_{t_0}\}(s)=\int_0^{\infty}e^{-st}\delta(t-t_0)dt=e^{-st_0}$$

When $t_0=0$ we write $\delta_0=\delta$ and we will agree that $\mathcal{L}\{\delta\}=1$.
We can compute the anti-derivative of δ_{t_0}

$$\int_{-\infty}^{t}\delta(t-t_0)dt=\begin{cases}1, & t>t_0\\ 0, & t<t_0\end{cases}=u_{t_0}(t)$$

Solving ODE with the Dirac function

Main idea: *business as usual*

Example (old exam problem):

$$y''(t) + 100y(t) = \delta(t-2),\ \ y(0) = 0, y'(0) = 0.$$

Test question: can you find $y(1)$ without solving the equation?

$$\mathcal{L} : y \mapsto Y(s),\ \ y'' \mapsto s^2 Y(s),\ \ \delta(t-2) \mapsto e^{-2s}$$

$$Y(s)(s^2 + 100) = e^{-2s};\ \ Y(s) = \frac{e^{-2s}}{s^2 + 100}.$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2 + 100}\right) = 0.1u(t-2)\sin 10(t-2).$$

Convolution

Let f and g be two piece-wise continuous functions on $[0, +\infty)$ we define a new function

$$h(t) = (f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

It is called the convolution of f and g. Basic rules for convolutions are

- $f * g = g * f$
- $f * (ag) = a(f * g)$ when a is a constant
- $f * (g_1 + g_2) = f * g_1 + f * g_2$
- $(f * g) * w = f * (g * w)$

Laplace transform: convolution theorem

Theorem

Suppose that f and g are piece-wise continuous functions and there Laplace transforms are defined when $s > a$, $\mathcal{L}\{f\} = F,\ \mathcal{L}\{g\} = G$. Then the Laplace transform of their convolution $f * g$ is also defined when $s > a$ and

$$\mathcal{L}\{f * g\}(s) = F(s)G(s)$$

Convolution: examples

1. $f(t) = t,\ g(t) = t^2,\ f * g = \int_0^t (t-\tau)\tau^2 d\tau = t^4/12$
   Check the convolution theorem
   $\mathcal{L}\{t\} = s^{-2},\ \mathcal{L}\{t^2\} = 2s^{-3}$ and $\mathcal{L}\{t^4/12\} = (24s^{-5}) = 2s^{-5}$.
2. $f(t) = \cos t,\ g(t) = 1,\ f * g = \int_0^t \cos\tau d\tau = \sin t$
   The Laplace transforms are:
   $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1},\ \mathcal{L}\{1\} = \frac{1}{s}$ and $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$
3. $g(t) = 1,\ f * g = \int_0^t f(\tau) d\tau$ and

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \frac{1}{s}\mathcal{L}\{f\}(s)$$

Applications of the convolution theorem, examples

Example

Find the Laplace transform of $f(t) = \int_0^t (t-\tau)^3 \sin(2\tau) d\tau$.

$$\mathcal{L}\{f\}(s) = \mathcal{L}\{t^3\}(s)\mathcal{L}\{\sin 2t\}(s) = \frac{6}{s^4} \cdot \frac{2}{s^2+4} = \frac{12}{s^6+4s^4}$$

Example

Find the inverse Laplace transform of $F(s) = \frac{1}{s^3 - s^2}$

$$\mathcal{L}^{-1}\{(s^3-s^2)^{-1}\} = \mathcal{L}^{-1}\{s^{-2}\}*\mathcal{L}^{-1}\{(s-1)^{-1}\} = t*e^t = \int_0^t e^\tau (t-\tau) d\tau$$

$$(= t(e^t-1) - (te^t - e^t + 1) = e^t - t - 1)$$