6.2/6.3 Laplace transform, examples and applications

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Laplace transform: definition and existence

A special machine which changes one function into another.
Input: $f(t)$, $t > 0$. Output:

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt$$

Definition

If f is a piece-wise continuous function on each interval $[0, A]$ then

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt = \lim_{A\to\infty} \int_0^A f(t)e^{-st}dt,$$

if the limit exists.

Theorem

If $|f(t)| < M\ e^{kt}$ (such f is called a function of exponential order), then $\mathcal{L}\{f\}(s)$ exists for all $s > k$.

Basic rules

- Linearity

$$\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}$$

- First shift rule

$$\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$$

- Second shift rule

$$\mathcal{L}\{u_c(t)f(t-c)\}(s) = e^{-cs}\mathcal{L}\{f\}(s), \quad c > 0$$

- Derivatives

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0),\ \mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0),$$

$$\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$

Laplace transform of derivatives

Suppose that f and its derivatives are continuous functions of exponential type. Then integration by parts gives

- $$\mathcal{L}\{f'\} = \int_0^\infty f' e^{-st} dt = f(t)e^{-st}\Big|_0^\infty -$$
  $$- \int_0^\infty f(t)(-se^{-st})dt = s\mathcal{L}\{f\} - f(0),$$
- $$\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0),$$
- $$\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0)$$

Laplace transform of integrals

If f is a piece-wise continuous function of exponential type then

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$$

The function $g(t) = \int_0^t f(\tau)d\tau$ is continuous,of exponential type and $g'(t) = f(t)$.

Applications to ODE

Solve the initial value problem

$$ay'' + by' + cy = g, \quad y(0) = K_0, \; y'(0) = K_1$$

Apply the Laplace transform

$$(as^2 + bs + c)Y - (as + b)K_0 - aK_1 = G$$

Then

$$Y(s) = \frac{G(s)}{as^2 + bs + c} + \frac{(as + b)K_0 + aK_1}{as^2 + bs + c}$$

We can find $\mathcal{L}\{(as^2 + bs + c)^{-1}\}$ and use the inverse Laplace transform to compute y.

Example 1

$$y'' - 9y = 1,\ y(0) = 1,\ y'(0) = 0$$

- Apply the Laplace transform $s^2 Y - s - 9Y = \frac{1}{s}$
- Solve for Y and use partial fractions to write down the answer

$$Y(s) = \frac{s^2+1}{s(s^2-9)} = -\frac{1}{9s} + \frac{0.5}{9(s-3)} + \frac{0.5}{9(s+3)}$$

- Find y by performing the inverse transform $y(t) = -1/9 + 1/18e^{3t} + 1/18e^{-3t}$

Example 2

$$y'' + y' - 2y = \sin t, \;\; y(0) = 0, y'(0) = 1$$

1.
$$s^2Y - 1 + sY - 2Y = \frac{1}{s^2+1}$$

2.
$$Y(s) = \frac{s^2+2}{(s^2+1)(s^2+s-2)}$$

We want to decompose it using partial fractions (see next slide):

$$Y(s) = -0.1\frac{s}{s^2+1} - 0.3\frac{1}{s^2+1} + 0.5\frac{1}{s-1} - 0.4\frac{1}{s+2}$$

3. Applying the inverse transform we get

$$y(t) = -0.1\cos t - 0.3\sin t + 0.5e^{t} - 0.4e^{-2t}$$

Partial fractions: example

We look for the representation

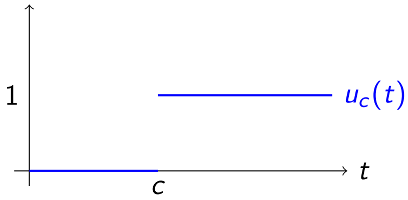
$$Y(s) = \frac{as+b}{s^2+1} + \frac{c}{s-1} + \frac{d}{s+2},$$

where
$s^2+2 = (as+b)(s-1)(s+2) + c(s^2+1)(s+2) + d(s^2+1)(s-1).$
Now we either open the brackets, compare the coefficients and solve a system of linear equations or we substitute $s = 1, 2, i$ and find the constants $a = -0.1$, $b = -0.3$, $c = 0.5$, $d = -0.4$.

Laplace transform of discontinuous functions

The building block for discontinuous functions is the step function (Heaviside's function) u_c:

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$



For $c \geq 0$ we compute its Laplace transform:

$$\mathcal{L}\{u_c\}(s) = \int_c^{\infty} e^{-st} dt = \frac{e^{-cs}}{s}, \ s > 0.$$

Further, the change of variables gives the second shift rule

$$\mathcal{L}\{u_c(t)f(t-c)\}(s) = e^{-cs}\mathcal{L}\{f\}(s), \quad c \geq 0.$$

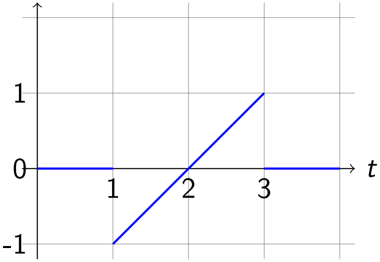
We use it to evaluate the Laplace transform of piecewise defined functions.

Example

Find the Laplace transform of the function

$$f(t) = \begin{cases} 0, & t < 1 \\ t-2, & 1 \le t \le 3 \\ 0, & t > 3 \end{cases}$$

First we look at the graph of this function



And rewrite the function as

$$\begin{aligned} f(t) = & \quad (t-2)(u_3(t) - u_1(t)) \\ = & \quad (t-3)u_3(t) + u_3(t) \\ & -(t-1)u_1(t) + u_1(t) \end{aligned}$$

Then, using the rules above, we compute

$$\mathcal{L}f(s) = \frac{e^{-3s}}{s^2} + \frac{e^{-3s}}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s}$$

Inverse Laplace transform: Example

An important step in the application of the Laplace transform to ODE is to find the inverse Laplace transform of the given function. Find $f(t)$ such that $\mathcal{L}\{f\} = F$ is

$$F(s) = \frac{e^{-2s}}{s^2 + 2s - 3}$$

First, using the partial functions

$$\frac{1}{s^2 + 2s - 3} = \frac{1}{4}\left(\frac{1}{s-1} - \frac{1}{s+3}\right).$$

Then we write

$$F(s) = \frac{1}{4}\left(\frac{e^{-2s}}{s-1} - \frac{e^{-2s}}{s+3}\right)$$

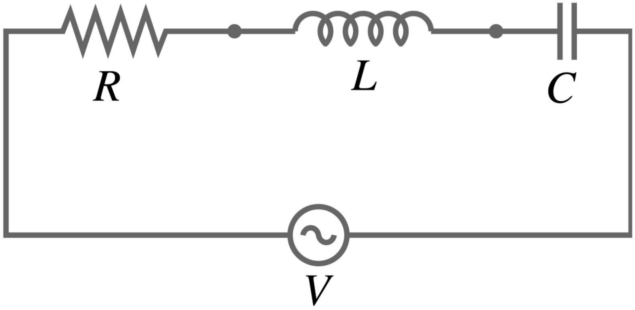
and using the second shift rule and the table to get

$$\mathcal{L}^{-1}(F)(t) = \frac{u_2(t)}{4}(e^{t-2} - e^{-3(t-2)})$$

RLC-circuit

Consider an RLC circuit consisting of a resistor R, inductor L, and capacitor C ,which is driven by a voltage source v.
Let q be the charge on the capacitor and let the current in the circuit be i. By Kirchhoff's voltage laws

$$Li'(t) + Ri(t) + \frac{1}{C}q(t) = v(t)$$



RLC-circuit cont.

$$Li'(t) + Ri(t) + \frac{1}{C}q(t) = v(t)$$

In this equation: resistance, inductance, capacitance and voltage are known quantities but current and charge are unknown quantities, $q(t) = \int_0^t i(\tau)d\tau$.
We apply the Laplace transform

$$L(sI(s) - i(0)) + RI(s) + \frac{1}{sC}I(s) = V(s)$$

RLC-circuit example

Find $i(t)$ in the circuit with

$R = 50.2\Omega, L = 1H, C = 0.1F, v(t) = 99.6(u(t) - u(t-3)), i(0) = 0$

After the Laplace transform we get

$$(s + 50.2 + 10/s)I(s) = 100(1 - e^{-3s})/s$$

$$I(s) = \frac{99.6(1 - e^{-3s})}{s^2 + 50.2s + 10} = \frac{99.6(1 - e^{-3s})}{49.8}\left(\frac{1}{s + 0.2} - \frac{1}{s + 50}\right)$$

Then

$$i(t) = 2(e^{-0.2t} - e^{-50t} - e^{-0.2(t-3)}u(t-3) + e^{-50(t-3)}u(t-3))$$