Lecture 26 More integrals

1. Fourier

Example: $$I(\omega) = \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{(x-i)(x-2i)}\,dx$$

We want the same procedure as for rational functions.

Q: Which half-plane (upper or low) to be chosen.

A: Look at the behavior of the function – it should decay!

$$f(z) = \frac{e^{i\omega z}}{(z-i)(z-2i)}$$

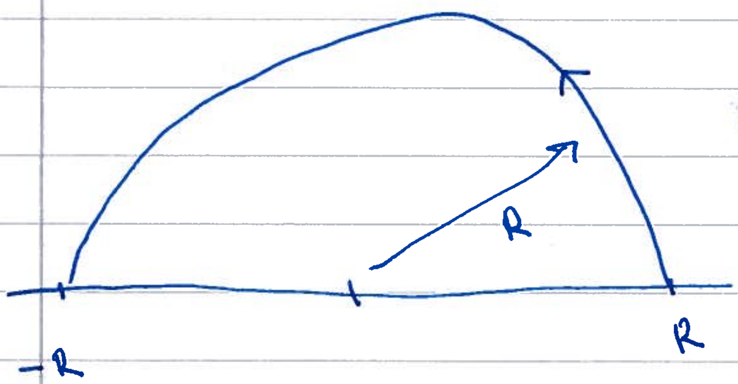
Let $\omega > 0$ and $z = x + iy$

$$|e^{i\omega z}| = e^{i\omega x - \omega y}$$

↑ decays as $y > 0$

$\Rightarrow$ we take upper half-plane.

$$I(w) = \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{iwx}}{(x-i)(x-2i)}\,dx$$



$$\Gamma_R = \{z : |z| = R, \ \mathrm{Im}\, z > 0\}$$

$$C_R = [-R, R] \cup \Gamma_R$$

$$\int_{C_R} \frac{e^{izw}}{(z-i)(z-2i)}\,dz = \int_{-R}^{R} \frac{e^{ixw}}{(x-i)(x-2i)}\,dx$$

$$+ \int_{\Gamma_R} \frac{e^{izw}}{(z-i)(z-2i)}\,dz$$

$$\int_{C_R} \frac{e^{izw}}{(z-i)(z-2i)}\,dz = 2i\pi\left(\mathrm{Res}_{i} + \mathrm{Res}_{2i}\right) \dots =$$

$$= 2\pi\left(e^{-2w} - e^{-w}\right)$$

So

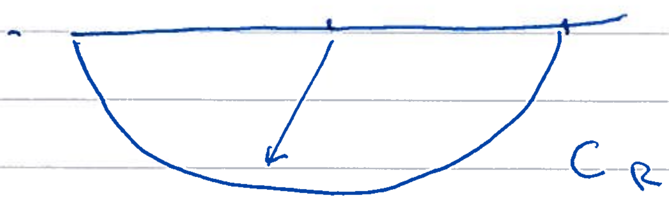
$$2\pi\left(e^{-2w}-e^{-w}\right)=\underbrace{\int_{-R}^{R}\frac{e^{iwx}}{(x-i)(x-2i)}\,dx}_{\xrightarrow{R\to\infty}\ I(w)}+\underbrace{\int_{\Gamma_R}\frac{e^{izw}}{(z-i)(z-2i)}\,dz}_{\xrightarrow{R\to\infty}\ 0}$$

<u>Finally</u> $I(w)=2\pi\left(e^{-2w}-e^{-w}\right)$ if $w>0$.

If <u>$w<0$</u>, $z=x+iy$,

$|e^{iwz}|=e^{-wy}$ decays as $y<0$

so one has to choose the lower half plane!



C_R

Function $e^{i\omega z}/(z-i)(z-2i)$

does not have singularities inside C_R

$$\Rightarrow \int_{C_R} \frac{e^{i\omega z}}{(z-i)(z-2i)}\,dz = 0$$

and $\quad I(-\omega) = 0$

__Finally:__

$$\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{(x-i)(x-2i)}\,dx = \begin{cases} 2\pi\left(e^{-2\omega} - e^{-\omega}\right) & \text{if } \omega > 0 \\ 0 & \text{if } \omega < 0. \end{cases}$$

__Question for you:__ What if $\omega = 0$?

More examples of Fourier integrals

- $$\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{x^2+4}\,dx$$

- $$\int_{-\infty}^{\infty} \frac{\cos \omega x}{(x-i)(x+2i)}\,dx$$

Additional integral

(if time permits)

$$\int_0^\infty \frac{x^{1/2}}{(x+1)(x+2)}\,dx$$

Exercise:

$$\int_0^\infty \frac{x^\alpha}{(x+1)(x+2)}\,dx$$

for any α, $0<\alpha<1$.