16.4 Residue Integration of real integrals

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Residue theorem

Theorem

If $f(z)$ is analytic in a domain D except for finite number of isolated singularities and C is a simple closed curved in D (with counterclockwise orientation) then

$$\oint f(z)dz = 2\pi i \sum_{j=1}^{k} \mathrm{Res}_{z_j} f(z),$$

where the sum is taken over all singular points enclosed by C.

Functions of $\cos\theta$ and $\sin\theta$

Many integrals of the form

$$J = \int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$$

may be considered as complex integrals of analytic functions (when F is an analytic function) over a unit circle $z = e^{i\theta}$. We have

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

Further, $dz = ie^{i\theta}d\theta$ and $d\theta = dz/(iz)$. Performing such a change of variables we reduce the integration to

$$J = \oint_{|z|=1} f(z)dz$$

which in case of analytic f can be computed by the residue method

Example: trigonometric integral

Example

$J = \int_0^{2\pi} \frac{\cos\theta}{2+\sin\theta} d\theta$. We have $\sin\theta = (z - z^{-1})/(2i)$ and

$$J = \oint_{|z|=1} \frac{i(z+z^{-1})}{4i + z - z^{-1}} \frac{dz}{iz} = \oint_{|z|=1} \frac{(z^2+1)}{z(z^2+4iz-1)} dz$$

The function $f(z) = \frac{(z^2+1)}{z(z^2+4iz-1)} = \frac{p(z)}{q(z)}$ is a rational function with poles in the zeros of the polynomial $q(z) = z(z^2+4iz-1)$. It has three simple zeros, $z_1 = 0$, $z_2 = i(-2+\sqrt{3})$ and $z_3 = i(-2-\sqrt{3})$. First two are inside the unit circle. Then

$$J = 2\pi i(\mathrm{Res}_{z_1} f(z) + \mathrm{Res}_{z_2} f(z)) = 2\pi i \left(\frac{p(z_1)}{q'(z_1)} + \frac{p(z_2)}{q'(z_2)} \right) = 0$$

One more example: an old exam problem

$$\int_0^{\pi} \frac{2}{3\sin 2x + 5} dx$$

We make a change of variable $2x = \theta$ and get

$$\int_0^{2}\pi \frac{1}{3\sin\theta + 5} d\theta = \oint_{|z|=1} \frac{2i}{3z - 3z^{-1} + 10i} \frac{dz}{iz}$$
$$= \oint_{|z|=1} \frac{2dz}{3z^2 + 10iz - 3}$$

We integrate the function with two poles $z_1 = -i/3$ and $z_2 = -3i$. Only one of them is in the unit disk. Then

$$\oint_{|z|=1} \frac{2dz}{3z^2 + 10iz - 3} dz = 2\pi i \mathrm{Res}_{z_1} \frac{2}{3z^2 + 10iz - 3} = \frac{4\pi i}{3(z_1 - z_2)} = \frac{\pi}{2}$$

Improper integral

Our second series of examples are integrals of the form

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx$$

Suppose that $f(z)$ is analytic in the upper half-plane except for a finite number of isolated singularities and $|f(z)| \leq k|z|^{-2}$ when $z = x + iy, y > 0$ and $|z| > R_0$. Then we have

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{z_j} \mathrm{Res}_{z_j} f(z)$$

where the sum is taken over all isolated singularities of f in the upper half-plane.

Examples

Example

$$\int_{-\infty}^{\infty} \frac{x+1}{x^4+1} dx = 2\pi i \sum_{z_j} \mathrm{Res}_{z_j} \frac{z+1}{z^4+1}$$

We look at the poles which are zeros of the equation $z^4 + 1 = 0$ and take only ones in the upper half-plane, they are

$$z_1 = e^{i\pi/4}, \quad z_2 = e^{3\pi/4}$$

We have $z_1 + z_2 = \sqrt{2}i,\ z_1^2 + z_2^2 = 0$ and

$$\mathrm{Res}_{z_j} \frac{z+1}{z^4+1} = \frac{z_j+1}{4z_j^3} = \frac{z_j^2+z_j}{4z_j^4} = \frac{-z_j^2 - z_j}{4}$$

Then $\int_{-\infty}^{\infty} \frac{x+1}{x^4+1} dx = \frac{\sqrt{2}\pi}{2}$