16.3 Residues

Eugenia Malinnikova, NTNU

November 10, 2016

Singularity at infinity

Let $f(z)$ be analytic in a domain $\{|z| > R\}$, we can define $g(w) = f(1/w)$ in the disk without the origin $\{z : 0 < |z| < 1/R\}$. We say that f has singularity (essential singularity, pole) at infinity if g has singularity (essential singularity, pole) at the origin.
We can also consider the Laurent series of f in the domain $|z| > R$,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

At infinity the principle part is the first series, if it contains non-constant terms then f has singularity at infinity,if only finitely many of a_n are non-zeros then f has a pole at infinity.

Residue

Let z_0 be an isolated singularity of an analytic function $f(z)$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

then b_1 is called the residue of f at z_0, $b_1 = \mathrm{Res}_{z_0} f(z)$.

Theorem

If f is analytic in $\{0 < |z - z_0| < r\}$ and C is a simple closed curve enclosing z_0 then

$$\oint_C f(z)dz = 2\pi i \mathrm{Res}_{z_0} f(z)$$

Examples

Example

If C is a simple closed curve enclosing the origin (with counterclockwise orientation) then

1. $$\oint_C \frac{1}{z} dz = 2\pi i, \quad \oint_C \frac{1}{z^m} dz = 0, \quad m = 2, 3, ...$$
2. $$\oint_C e^{1/z} dz = 2\pi i \mathrm{Res}_0 e^{1/z} = 2\pi i,$$
   since $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2z^2} + ...$
3. $$\oint_C \frac{\cos z}{z^2} dz = 2\pi i \mathrm{Res}_0 \frac{\cos z}{z^2} = 0$$

Formulas for residues

Simple pole at z_0: If $f(z) = \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \ldots$ then

$$\mathrm{Res}_{z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z)$$

If $f(z) = \frac{p(z)}{q(z)}$, where $p(z_0) \neq 0$ and q has zero of order one at z_0,then f has a simple pole at z_0 and

$$\mathrm{Res}_{z_0} f(z) = \frac{p(z_0)}{q'(z_0)}$$

Higher order pole: If f has a pole of order m at z_0 then

$$\mathrm{Res}_{z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z)\right)$$

Residue theorem

Theorem

If $f(z)$ is analytic in a domain D except for finite number of isolated singularities and C is a simple closed curved in D (with counterclockwise orientation) then

$$\oint f(z)dz = 2\pi i \sum_{j=1}^{k} \mathrm{Res} z_j f(z),$$

where the sum is taken over all singular points enclosed by C.

Examples

Let C be the circle of radius 2 centered at the origin. Find all singularities of the following functions and compute the integral

$$\oint_C f(z)dz$$

1. $f(z) = \frac{2+z}{z^3+z^2}$
2. $f(z) = \frac{\tan \pi z}{z^2-1}$
3. $f(z) = \sin \frac{1}{z}$
4. $f(z) = e^{z^2+\frac{1}{z^2}}$