16.1 Laurent series
16.2 Zeros and singular points

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Taylor series

**Theorem**

Let $f(z)$ be analytic in a domain $D$ and $z_0$ be any point in $D$. There is precisely one Taylor series with center $z_0$ that represents $f(z)$. The disk of convergence for this series is the largest disk centered at $z_0$ where $f(z)$ is analytic.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$
Theorem

Suppose that $f$ is analytic in a domain $D$ which contains a circular ring, $A = \{ r_0 \leq |z - z_0| \leq R_0 \}$. Then $f$ can be represented by the Laurent series with center at $z_0$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

in this ring. The series converges in some domain $\{ r < |z - z_0| < R \}$ with $r \leq r_0$ and $R \geq R_0$.

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta, \quad b_n = \frac{1}{2\pi i} \oint_C f(\zeta)(\zeta - z_0)^{n-1} d\zeta,$$

where $C$ is any closed curve in $D$ that encloses $z_0$. 
Example

Example

\[ f(z) = \frac{1}{1-z}, \; z_0 = 0 \]

1. This function is analytic in the disc \( |z| < 1 \) and has the Taylor series expansion

\[ f(z) = 1 + z + z^2 + \ldots = \sum_{n=0}^{\infty} z^n \]

2. It is also analytic in the domain \( |z| > 1 \) and has the Laurent series expansion in that domain. We write

\[ f(z) = \frac{1}{1-z} = -\frac{1}{z(1-z^{-1})} = -z^{-1}(1+z^{-1}+\ldots) = \sum_{n=1}^{\infty} -\frac{1}{z^n} \]
More examples

- $z^{-2}e^z = \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^n}{(n+2)!}$ for $|z| > 0$

- $e^{1/z} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!z^n}$ for $|z| > 0$

- $f(z) = \frac{1}{(z + 1)(z + 2)} = \frac{1}{z + 1} - \frac{1}{z + 2}$

It has Taylor series expansion in $\{|z| < 1\}$, a Laurent series expansion in $\{1 < |z| < 2\}$, and another Laurent series expansion in $\{|z| > 2\}$. 
Singular points

**Definition**

We say that a function $f$ has an isolated singularity at some point $z_0$ if $f$ is analytic in $\{0 < |z - z_0| < r\}$ for some $r > 0$ but not analytic at $z_0$.

If $f$ has an isolated singularity at $z_0$ then it has Laurent series expansion in $0 < |z - z_0| < r$ which contains negative powers

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

The second series is called the principle part of the Laurent series.

If the principle part is finite $\frac{b_1}{(z-z_0)} + \ldots + \frac{b_m}{(z-z_0)^m}$ with $b_m \neq 0$ then we say that $f$ has a pole at $z_0$ of order $m$. If the principal part is not finite, we say that $z_0$ is an essential singularity of $f$. 
If \( f(z) \) is analytic in \( |z - z_0| < r \) and \( f \) has a zero of order \( d \) at \( z_0 \) then \( 1/f \) has a pole of order \( d \) at zero

\[
f(z) = (z - z_0)^d g(z)
\]

\[
\frac{1}{f(z)} = (z - z_0)^{-d} h(z) = \sum \frac{h(z_0)}{(z - z_0)^d} + \frac{h'(z_0)}{(z - z_0)^{d-1}} + \ldots
\]

where \( h(z) = 1/g(z) \) is analytic in some disk centered at \( z_0 \) and \( h(z_0) \neq 0 \).

If \( f \) has a pole at \( z_0 \) then \( |f(z)| \to \infty \) as \( z \to z_0 \).
Example: $f(z) = \sin \frac{1}{z}$ has essential singularity at the origin.

$$f(z) = \frac{1}{z} - \frac{1}{6z^3} + \frac{1}{5!z^5} - \ldots$$

Theorem (Picard)

*Suppose that $f$ has an essential singularity at $z_0$. Then there exists a complex number $c_0$ such that for any $c \neq c_0$ the equation $f(z) = c$ has solutions in each disk centered at $z_0$.***
Behavior near a point

Suppose that $f(z)$ is analytic in $0 < |z - z_0| < r$ and $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ Then the following situations may occur:

1. $f(z)$ is bounded in $0 < |z - z_0| < r_0$ for some $r_0$. Then all $b_n = 0$ and we can extend $f$ to an analytic function in $|z - z_0| < r$ by defining $f(z_0) = a_0$. We say that $f$ has a removable singularity at $z_0$.

2. $|f(z)| \to \infty$ as $z \to z_0$. Then only finitely many of $b_n$ are non-zeros and $f$ has a pole at $z_0$.

3. $f(z)$ is unbounded but $|f| \not\to \infty$ then the principal part of the Laurent series has infinitely many non-zero terms and $f$ has an essential singularity at $z_0$. 