16.1 Laurent series
16.2 Zeros and singular points

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November 7, 2016

Taylor series

Theorem

Let $f(z)$ be analytic in a domain D and z_0 be any point in D. There there is precisely one Taylor series with center z_0 that represents $f(z)$. The disk of convergence for this series is the largest disk centered at z_0 where $f(z)$ is analytic.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

Laurent series

Theorem

Suppose that f is analytic in a domain D which contains a circular ring, $A = \{r_0 \leq |z - z_0| \leq R_0\}$. Then f can be represented by the Laurent series with center at z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

in this ring. The series converges in some domain $\{r < |z - z_0| < R\}$ with $r \leq r_0$ and $R \geq R_0$.

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta, \quad b_n = \frac{1}{2\pi i} \oint_C f(\zeta)(\zeta - z_0)^{n-1} d\zeta,$$

where C is any closed curve in D that encloses z_0.

Example

Example

$f(z) = \frac{1}{1-z},\ z_0 = 0$

1. This function is analytic in the disc $|z| < 1$ and has the Taylor series expansion

$$f(z) = 1 + z + z^2 + ... = \sum_{n=0}^{\infty} z^n$$

2. It is also analytic in the domain $|z| > 1$ and has the Laurent series expansion in that domain. We write

$$f(z) = \frac{1}{1-z} = -\frac{1}{z(1-z^{-1})} = -z^{-1}(1+z^{-1}+...) = \sum_{n=1}^{\infty} -\frac{1}{z^n}$$

More examples

- $z^{-2}e^z = \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^n}{(n+2)!}$ for $|z| > 0$
- $e^{1/z} = 1 + \sum_{n=1}^{\infty} \frac{1}{n! z^n}$ for $|z| > 0$
- $$f(z) = \frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2}$$

  it has Taylor series expansion in $\{|z| < 1\}$, a Laurent series expansion in $\{1 < |z| < 2\}$, and another Laurent series expansion in $\{|z| > 2\}$.

Singular points

Definition

We say that a function f has an isolated singularity at some point z_0 if f is analytic in $\{0 < |z - z_0| < r\}$ for some $r > 0$ but not analytic at z_0.

If f has an isolated singularity at z_0 then it has Laurent series expansion in $0 < |z - z_0| < r$ which contains negative powers

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

The second series is called the principle part of the Laurent series. If the principle part is finite $\frac{b_1}{(z-z_0)} + ... + \frac{b_m}{(z-z_0)^m}$ with $b_m \neq 0$ then we say that f has a pole at z_0 of order m. If the principal part is not finite, we say that z_0 is an essential singularity of f.

Poles

If $f(z)$ is analytic in $|z - z_0| < r$ and f has a zero of order d at z_0 then $1/f$ has a pole of order d at zero

$$f(z) = (z - z_0)^d g(z)$$

$$\frac{1}{f(z)} = (z - z_0)^{-d} h(z) = \sum \frac{h(z_0)}{(z - z_0)^d} + \frac{h'(z_0)}{(z - z_0)^{d-1}} + \dots$$

where $h(z) = 1/g(z)$ is analytic in some disk centered at z_0 and $h(z_0) \neq 0$.
If f has a pole at z_0 then $|f(z)| \to \infty$ as $z \to z_0$.

Essential singularity

Example: $f(z) = \sin\frac{1}{z}$ has essential singularity at the origin.

$$f(z) = \frac{1}{z} - \frac{1}{6z^3} + \frac{1}{5!z^5} - \dots$$

Theorem (Picard)

Suppose that f has an essential singularity at z_0. Then there exists a complex number c_0 such that for any $c \neq c_0$ the equation $f(z) = c$ has solutions is each disk centered at z_0.

Behavior near a point

Suppose that $f(z)$ is analytic in $0 < |z - z_0| < r$ and $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ Then the following situations may occur:

1. $f(z)$ is bounded in $0 < |z - z_0| < r_0$ for some r_0.Then all $b_n = 0$ and we can extend f to an analytic function in $|z - z_0| < r$ by defining $f(z_0) = a_0$. We say that f has a removable singularity at z_0.
2. $|f(z)| \to \infty$ as $z \to z_0$. Then only finitely many of b_n are non-zeros and f has a pole at z_0.
3. $f(z)$ is unbounded but $|f| \not\to \infty$ then the principal part of the Laurent series has infinitely many non-zero terms and f has an essential singularity at z_0.