15.3 Functions given by power series.
15.4 Taylor series.

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Power series

We will study functions represented by power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

z_0 - center of the power series.

- The series converges in a disk centered at z_0, $\{z : |z - z_0| < R\}$, $0 \leq R \leq +\infty$ (this disk may degenerate to a single point z_0 or be the whole plane)
- The series diverges outside the closed disk, i.e. it diverges when $|z - z_0| > R$. We don't know the behavior on the circle $\{z : |z - z_0| = R\}$, it depends on the coefficients.
- Formula for the radius of convergence is

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \text{ if the limit exists,}$$

Examples

1. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges everywhere, $R = \infty$, as for real numbers we have $\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$.
2. $\sum_{n=0}^{\infty} z^n$ converges when $|z| < 1$ to the sum $\frac{1}{1-z}$ and diverges when $|z| \geq 1$, $R = 1$.
3. $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges when $|z| \leq 1$ and diverges when $|z| > 1$, $R = 1$
4. $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges when $|z| < 1$ and diverges when $|z| > 1$, $R = 1$; when $z = 1$ the series diverges, when $z = -1$ it converges. (What happens for other z on the unit circle?)
5. $\sum_{n=0}^{\infty} n! z^n$ diverges when $z \neq 0$, $R = 0$ (use the formula for R).
6. $\sum_{n=0}^{\infty} 2^n (z+i)^{2n}$ converges when $|z+i| < 1/\sqrt{2}$, $R = 1/\sqrt{2}$.
7. $\sum_{n=0}^{\infty} n(z-1)^n$ converges when $|z-1| < 1$, $R = 1$.

The sum of power series

Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series with positive radius of convergence R. Then for each $z : |z-z_0| < R$ the sum

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

is defined. We say that f is represented by the power series.

Theorem

If f is represented by a power series in some disk $\{z : |z-z_0| < R\}$ with $R > 0$ then f is continuous at z_0.

Uniqueness of power series and operation

Theorem

Suppose that $f(z)$ is represented by two series $\sum a_n(z-z_0)^n$ and $\sum_n b_n(z-z_0)^n$ in some disk centered at z_0. Then these series are equal, $a_n = b_n$.

Let $f(z) = \sum a_n(z-z_0)^n$ and $g(z) = \sum_n b_n(z-z_0)^n$ and both radii of convergence are $\geq R$. Then

- The series $\sum_n (a_n + b_n)(z-z_0)^n$ converges in $\{|z-z_0| < R\}$ to $f(z) + g(z)$
- For $\{z : |z-z_0| < R\}$ the product $f(z)g(z)$ is represented by the series

$$a_0b_0 + (a_0b_1 + b_0a_1)(z-z_0) + (a_0b_2 + a_1b_1 + a_2b_0)(z-z_0)^2 + \ldots =$$

$$= \sum_{n=0}^{\infty} (a_0b_n + a_1b_{n-1} + \ldots + a_nb_0)z^n$$

Term-wise differentiation and integration

Consider the following power series

1. $\sum_{n=0}^{\infty} a_n(z - z_0)^n$
2. $\sum_{n=0}^{\infty} na_n(z - z_0)^{n-1}$
3. $\sum_{n=1}^{\infty} \frac{a_n}{n+1}(z - z_0)^{n+1}$

These series has the same radius of convergence. The second one is obtained from the first by term-wise differentiation. The third is the term-wise integration of the first.

Theorem

If $f(z)$ is represented by a power series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ with a positive radius of convergence R, then $f(z)$ is analytic in the open disk $\{z : |z - z_0| < R\}$ and $f'(z)$ is represented by the series $\sum_{n=1}^{\infty} na_n(z - z_0)^{n-1}$.

Term-wise integration

Theorem

If $f(z)$ is represented by a power series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ with a positive radius of convergence R, then the anti-derivative $F(z)$ is represented by the series $C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - z_0)^{n+1}$.

Example

If $f(z) = \sum_0^{\infty} z^n = \frac{1}{1-z}$ then

$$\log(1 - z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}, \quad |z| < 1$$

Examples

1. $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ and $(e^z)' = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!} = e^z$.
2. $(1-z)^{-1} = \sum_{n=0}^{\infty} z^n$ and $(1-z)^{-2} = \sum_{n=1}^{\infty} nz^n$.
3. $\sum_{n=1}^{\infty} \frac{(z-i)^n}{n3^n}$ converges to an analytic function f in $|z-i|<3$ and $f'(z) = \sum_{n=1}^{\infty} \frac{(z-i)^{n-1}}{3^n} = \frac{1}{3(1-(z-i)/3)} = \frac{3}{3+i-z}$. Then
$$f(z) = -3\ln(z-3+i)$$
when $|z-i|<3$
4. $\sum_{n=0}^{\infty} n(n-1)(z-\pi i)^n$, the radius of convergence is $R=1$ and
$$f(z) = \sum_{n=0}^{\infty} n(n-1)(z-\pi i)^n = (z-\pi i)^2 \sum_{n=0}^{\infty} n(n-1)(z-\pi i)^{n-2}$$
$$= (z-\pi i)^2 \left(\sum_{n=0}^{\infty} (z-\pi i)^n\right)'' = (z-\pi i)^2 \left(\frac{1}{1+\pi i - z}\right)'' =$$
$$= \frac{2(z-\pi i)^2}{(1+\pi i - z)^3}$$

Power series: example

Geometric series: $z \in \mathbb{C}$ and $w_n = z^n$

$$\sum_0^\infty z^n = \begin{cases} \frac{1}{1-z}, & |z| < 1, \\ \text{diverges}, & |z| \geq 1. \end{cases}$$

Expansion of the Cauchy kernel:
Fix $z_0 \in \mathbb{C}$ and let $\zeta, z \in \mathbb{C}$ be such that $|z - z_0| < |\zeta - z_0|$. Then

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} =$$

$$\frac{1}{\zeta - z_0} \sum_{n=0}^\infty \left(\frac{z - z_0}{\zeta - z_0} \right)^n = \sum_{n=0}^\infty \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}$$

We are going to use this formula for making expansions of analytic functions into power series !

Expansions of analytic functions into power series

$z_0 \in \mathbb{C}$, $R > 0$, $f(z)$ is analytic in $\{z : |z - z_0| < R\}$.

Cauchy representation + expansion of the Cauchy kernel $\Rightarrow$

$$f(z) = \frac{1}{2i\pi} \int_{|\zeta - z_0| = R} \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$= \frac{1}{2i\pi} \int_{|\zeta - z_0| = R} f(\zeta) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} d\zeta =$$

$$\sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2i\pi} \int_{|\zeta - z_0| = R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

Expansions of analytic functions into power series

$$f(z) = \frac{1}{2i\pi}\int_{|\zeta - z_0| = R} \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$= \frac{1}{2i\pi}\int_{|\zeta - z_0| = R} f(\zeta) \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\zeta - z_0)^{n+1}} d\zeta =$$

$$\sum_{n=0}^{\infty} (z-z_0)^n \underbrace{\frac{1}{2i\pi}\int_{|\zeta - z_0| = R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta}_{\frac{1}{n!} f^{(n)}(z_0)} =$$

$$= \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} f^{(n)}(z_0).$$

We obtained Taylor series expansion for analytic function. It converges <u>for all</u> z such that $|z - z_0| < R$.

Taylor's formula with remainder

If f is analytic in a disk $\{|z-z_0|<R\}$ then $f$can be represented by a power series

$$f(z)=\sum_{n=}^{\infty}a_nz^n$$

in this disk and $a_n=\frac{f^{(n)}(z_0)}{n!}$. We call it the Taylor series of f.

Taking partial sums of the Taylor series we obtain polynomial approximation of an analytic function

$$f(z)=f(z_0)+(z-z_0)f'(z_0)+\ldots+\frac{(z-z_0)^n}{n!}f^{(n)}(z_0)+R_n(z)$$

The remainder is given by the formula

$$R_n(z)=\frac{(z-z_0)^{n+1}}{2\pi i}\oint_{|\zeta-z_0|=R}\frac{f(\zeta}{(\zeta-z_0)^{n+1}(\zeta-z)}d\zeta$$

Taylor series as power series

Theorem

Let $f(z)$ be analytic in a domain D and z_0 be any point in D. There there is precisely one Taylor series with center z_0 that represents $f(z)$. The disk of convergence for this series is the largest disk centered at z_0 where $f(z)$ is analytic.

Example

1. $f(z) = \frac{1}{1+z^2}$, $z_0 = 0$ the Taylor series converges for $|z| < 1$
2. $f(z) = (\cos z)^{-1}$, $z_0 = 0$ the Taylor series converges for $|z| < \pi/2$.

A power series with a nonzero radius of convergence is the Taylor series of its sum.