15.3 Functions given by power series.
15.4 Taylor series.

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We will study functions represented by power series

\[ f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \]

\( z_0 \) - center of the power series.

- The series converges in a disk centered at \( z_0 \),
  \( \{z : |z - z_0| < R\} \), \( 0 \leq R \leq +\infty \) (this disk may degenerate to a single point \( z_0 \) or be the whole plane)

- The series diverges outside the closed disk, i.e. it diverges when \( |z - z_0| > R \). We don’t know the behavior on the circle \( \{z : |z - z_0| = R\} \), it depends on the coefficients.

- Formula for the radius of convergence is
  \[ R = \lim_{n \to \infty} \frac{|a_n|}{|a_{n+1}|} \] if the limit exists,
Examples

1. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges everywhere, $R = \infty$, as for real numbers we have $\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$.

2. $\sum_{n=0}^{\infty} z^n$ converges when $|z| < 1$ to the sum $\frac{1}{1-z}$ and diverges when $|z| \geq 1$, $R = 1$.

3. $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges when $|z| \leq 1$ and diverges when $|z| > 1$, $R = 1$.

4. $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges when $|z| < 1$ and diverges when $|z| > 1$, $R = 1$; when $z = 1$ the series diverges, when $z = -1$ it converges. (What happens for other $z$ on the unit circle?)

5. $\sum_{n=0}^{\infty} n!z^n$ diverges when $z \neq 0$, $R = 0$ (use the formula for $R$).

6. $\sum_{n=0}^{\infty} 2^n(z + i)^{2n}$ converges when $|z + i| < 1/\sqrt{2}$, $R = 1/\sqrt{2}$.

7. $\sum_{n=0}^{\infty} n(z - 1)^n$ converges when $|z - 1| < 1$, $R = 1$. 
Let \( \sum_{n=0}^{\infty} a_n(z - z_0)^n \) be a power series with positive radius of convergence \( R \). Then for each \( z : |z - z_0| < R \) the sum

\[
f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n
\]

is defined. We say that \( f \) is represented by the power series.

**Theorem**

*If \( f \) is represented by a power series in some disk \( \{ z : |z - z_0| < R \} \) with \( R > 0 \) then \( f \) is continuous at \( z_0 \).*
Uniqueness of power series and operation

Theorem

Suppose that \( f(z) \) is represented by two series \( \sum a_n(z - z_0)^n \) and \( \sum b_n(z - z_0)^n \) in some disk centered at \( z_0 \). Then these series are equal, \( a_n = b_n \).

Let \( f(z) = \sum a_n(z - z_0)^n \) and \( g(z) = \sum b_n(z - z_0)^n \) and both radii of convergence are \( \geq R \). Then

- The series \( \sum_n(a_n + b_n)(z - z_0)^n \) converges in \( \{ |z - z_0| < R \} \) to \( f(z) + g(z) \)
- For \( \{ z : |z - z_0| < R \} \) the product \( f(z)g(z) \) is represented by the series

\[
a_0b_0 + (a_0b_1 + b_0a_1)(z-z_0) + (a_0b_2 + a_1b_1 + a_2b_0)(z-z_0)^2 + \ldots = \sum_{n=0}^{\infty} (a_0b_n + a_1b_{n-1} + \ldots + a_nb_0)z^n
\]
Consider the following power series

1. $\sum_{n=0}^{\infty} a_n (z - z_0)^n$
2. $\sum_{n=0}^{\infty} na_n (z - z_0)^{n-1}$
3. $\sum_{n=1}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$

These series has the same radius of convergence. The second one is obtained from the first by term-wise differentiation. The third is the term-wise integration of the first.

**Theorem**

If $f(z)$ is represented by a power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ with a positive radius of convergence $R$, then $f(z)$ is analytic in the open disk $\{z : |z - z_0| < R\}$ and $f'(z)$ is represented by the series $\sum_{n=1}^{\infty} na_n (z - z_0)^{n-1}$. 
Term-wise integration

Theorem
If $f(z)$ is represented by a power series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ with a positive radius of convergence $R$, then the anti-derivative $F(z)$ is represented by the series $C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - z_0)^{n+1}$.

Example
If $f(z) = \sum_{0}^{\infty} z^n = \frac{1}{1-z}$ then

$$\log(1 - z) = - \sum_{n=1}^{\infty} \frac{z^n}{n}, \quad |z| < 1$$
Examples

1. \( e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \) and \( (e^z)' = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!} = e^z \).

2. \( (1 - z)^{-1} = \sum_{n=0}^{\infty} z^n \) and \( (1 - z)^{-2} = \sum_{n=1}^{\infty} nz^n \).

3. \( \sum_{n=1}^{\infty} \frac{(z-i)^n}{n3^n} \) converges to an analytic function \( f \) in \(|z - i| < 3\) and \( f'(z) = \sum_{n=1}^{\infty} \frac{(z-i)^{n-1}}{3^n} = \frac{1}{3(1-(z-i)/3)} = \frac{3}{3+i-z} \). Then

\[
    f(z) = -3 \ln(z - 3 + i)
\]

when \(|z - i| < 3\)

4. \( \sum_{n=0}^{\infty} n(n-1)(z-\pi i)^n \), the radius of convergence is \( R = 1 \) and

\[
    f(z) = \sum_{n=0}^{\infty} n(n-1)(z-\pi i)^n = (z-\pi i)^2 \sum_{n=0}^{\infty} n(n-1)(z-\pi i)^{n-2}
\]

\[
    = (z-\pi i)^2 \left( \sum_{n=0}^{\infty} (z-\pi i)^n \right)'' = (z-\pi i)^2 \left( \frac{1}{1+\pi i-z} \right)'' =
\]

\[
    = \frac{2(z-\pi i)^2}{(1+\pi i-z)^3}
\]
Power series: example

Geometric series: \( z \in \mathbb{C} \) and \( w_n = z^n \)

\[
\sum_{0}^{\infty} z^n = \begin{cases} 
\frac{1}{1-z}, & |z| < 1, \\
diverges, & |z| \geq 1.
\end{cases}
\]

Expansion of the Cauchy kernel:
Fix \( z_0 \in \mathbb{C} \) and let \( \zeta, z \in \mathbb{C} \) be such that \( |z - z_0| < |\zeta - z_0| \). Then

\[
\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left( \frac{z - z_0}{\zeta - z_0} \right)^n = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}
\]

We are going to use this formula for making expansions of analytic functions into power series!
$z_0 \in \mathbb{C}, \ R > 0, \ f(z) \text{ is analytic in } \{z : |z - z_0| < R\}.$

Cauchy representation + expansion of the Cauchy kernel $\Rightarrow$

\[
f(z) = \frac{1}{2i\pi} \int_{|\zeta - z_0| = R} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2i\pi} \int_{|\zeta - z_0| = R} f(\zeta) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} d\zeta = \sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2i\pi} \int_{|\zeta - z_0| = R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta
\]
Expansions of analytic functions into power series

\[ f(z) = \frac{1}{2i\pi} \int_{|\zeta - z_0| = R} f(\zeta) \frac{d\zeta}{\zeta - z} = \frac{1}{2i\pi} \int_{|\zeta - z_0| = R} f(\zeta) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} d\zeta = \frac{1}{2i\pi} \int_{|\zeta - z_0| = R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)\]

\[ = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{n!} f^{(n)}(z_0).\]

We obtained Taylor series expansion for analytic function. It converges for all \( z \) such that \( |z - z_0| < R \).
Taylor’s formula with remainder

If $f$ is analytic in a disk $\{|z - z_0| < R\}$ then $f$ can be represented by a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

in this disk and $a_n = \frac{f^{(n)}(z_0)}{n!}$. We call it the Taylor series of $f$.

Taking partial sums of the Taylor series we obtain polynomial approximation of an analytic function

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \ldots + \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + R_n(z)$$

The remainder is given by the formula

$$R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \oint_{|\zeta - z_0| = R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}(\zeta - z)} d\zeta$$
Theorem

Let $f(z)$ be analytic in a domain $D$ and $z_0$ be any point in $D$. There is precisely one Taylor series with center $z_0$ that represents $f(z)$. The disk of convergence for this series is the largest disk centered at $z_0$ where $f(z)$ is analytic.

Example

1. $f(z) = \frac{1}{1+z^2}$, $z_0 = 0$ the Taylor series converges for $|z| < 1$
2. $f(z) = (\cos z)^{-1}$, $z_0 = 0$ the Taylor series converges for $|z| < \pi/2$.

A power series with a nonzero radius of convergence is the Taylor series of its sum.