15.1-2 Complex power series

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Series of complex numbers

Given a sequence $\{w_n\}_{n=1}^{\infty} \in \mathbb{C}$ consider *the series* $\sum_{n=1}^{\infty} w_n$ The series is convergent if its partial sums have a limit:

$$S_N = \sum_{n=1}^{N} w_n \ \to S, \text{ as } N \to \infty.$$

This limit is called the sum of the series.
Otherwise the series is divergent.

Relation to series of real numbers:
Let $w_n = u_n + iv_n$. The series $\sum w_n$ converges if and only if each of the real series $\sum u_n$, $\sum v_n$ converges.

Basic definitions and facts

- The series $\sum w_n$ is absolutely convergent if $\sum |w_n|$ is convergent.
- The series $\sum w_n$ is convergent $\Rightarrow$ $w_n \to 0$ as $n \to \infty$

Some sufficient conditions for convergence

- Cauchy criterion: $\sum_M^N w_n \to 0$ as $M, N \to \infty \Leftrightarrow \sum w_n$ converges
- Majorization: $a_n > 0,\ n = 1, 2, \ldots,\ |w_n| \leq a_n$ and $\sum a_n$ converges $\Rightarrow \sum w_n$ converges;
- Ratio test: $|w_{n+1}|/|w_n| \leq q < 1\ n = 1, 2,\ \ldots \Rightarrow \sum w_n$ converges;
- Root test: $(|w_n|)^{1/n} \leq q < 1\ n = 1, 2,\ \ldots \Rightarrow \sum w_n$ converges

The most important example

Geometric series: $z \in \mathbb{C}$ and $w_n = z^n$

$$\sum_0^\infty z^n = \begin{cases} \frac{1}{1-z}, & |z| < 1, \\ \text{diverges}, & |z| \geq 1. \end{cases}$$

Expansion of the Cauchy kernel:
Fix $z_0 \in \mathbb{C}$ and let $\zeta, z \in \mathbb{C}$ be such that $|z - z_0| < |\zeta - z_0|$. Then

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} =$$

$$\frac{1}{\zeta - z_0} \sum_{n=0}^\infty \left(\frac{z - z_0}{\zeta - z_0} \right)^n = \sum_{n=0}^\infty \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}$$

We are going to use this formula for making expansions of analytic functions into power series !

Further examples

1. $\sum_{n=1}^{\infty} \frac{(1+i)^n}{n!}$ converges (ratio test)
2. $\sum_{n=1}^{\infty} \frac{(1+i)^{2n}}{2^n}$ diverges , $|(1+i)^{2n}/2^n| = 1 \not\to 0$
3. $\sum_{n=1}^{\infty} \frac{n+i}{n^2}$ diverges, the real parts are $1/n$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges
4. $\sum_{n=1}^{\infty} \frac{i^n}{\sqrt{n}}$ converges,

$$\sum_{n=1}^{\infty} \frac{i^n}{\sqrt{n}} = \frac{i}{1} - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots$$

Separating real and imaginary parts we get two real alternating series, both of them converge and then the series converge. It does not converge absolutely, $\sum_n \left| \frac{i^n}{\sqrt{n}} \right|$ diverges.

Power series

We will study the functions represented by power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

z_0 - center of the convergence.

- The series converges in a disk centered at z_0, (this disk may degenerate to a single z_0 or be the whole plane)
- Formula for the radii R of the disk of convergence

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \text{ if the limit exists,}$$

- The series diverges outside the closed disk, we don't know the behavior on the circle

Examples

1. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges everywhere, $R = \infty$, as for real numbers we have $\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$.
2. $\sum_{n=0}^{\infty} z^n$ converges when $|z| < 1$ to the sum $\frac{1}{1-z}$ and diverges when $|z| \geq 1$, $R = 1$.
3. $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges when $|z| \leq 1$ and diverges when $|z| > 1$, $R = 1$
4. $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges when $|z| < 1$ and diverges when $|z| > 1$, $R = 1$; when $z = 1$ the series diverges, when $z = -1$ it converges. (What happens for other z on the unit circle?)
5. $\sum_{n=0}^{\infty} n! z^n$ diverges when $z \neq 0$, $R = 0$ (use the formula for R).
6. $\sum_{n=0}^{\infty} 2^n (z+i)^{2n}$ converges when $|z+i| < 1/\sqrt{2}$, $R = 1/\sqrt{2}$.
7. $\sum_{n=0}^{\infty} n(z-1)^n$ converges when $|z-1| < 1$, $R = 1$.