6.1 Laplace transform, introduction

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Laplace transform as an important engineering tool

In applications we have to handle discontinuous external forces (electrical switch, impulse). The mathematical approximation is *piecewise continuous functions*.

**Definition**
A function $f$ is said to be piecewise continuous on an interval $[a, b]$ if this interval can be partitioned into a finite number of intervals such that on each small open interval $f$ is continuous and $f$ has finite one-sided limits on the ends of these sub-intervals.

**Example**
The Heaviside function

$$u_0(t) = \begin{cases} 
0, & t < 0 \\
1, & t \geq 0
\end{cases} \quad u_c(t) = \begin{cases} 
0, & t < c \\
1, & t \geq c
\end{cases}$$
Integration of piece-wise continuous functions and improper integrals

If $f$ is a piece-wise continuous function on a finite interval $[a, b]$ then the integral $\int_a^b f(t)dt$ is defined as:

$$\int_a^b f(t)dt = \int_a^{t_1} f(t)dt + \int_{t_1}^{t_2} f(t)dt + ... + \int_{t_n}^b f(t)dt$$

it is the sum of integrals over sub-intervals on which $f$ is continuous.

Now suppose that $f$ is piece-wise continuous on $[a, A]$ for any $A > a$. Then we consider

$$\int_a^\infty f(t)dt = \lim_{A \to \infty} \int_a^A f(t)dt$$

if the limit exists (we say that the integral converges).
Examples and a comparison theorem

Example

Divergent integrals:

\[ \int_0^\infty e^{at} \, dt, \quad a \geq 0, \quad \int_1^\infty t^p \, dt, \quad p \geq -1, \quad \int_0^\infty \sin t \, dt \]

Convergent integrals:

\[ \int_0^\infty e^{at} \, dt, \quad a < 0, \quad \int_1^\infty t^p, \quad p < -1, \quad \int_0^\infty \frac{\sin t}{t} \, dt \]

Theorem

If \( \int_0^\infty g(t) \, dt \) converges and \( |f(t)| < g(t) \) then \( \int_0^\infty f(t) \, dt \) also converges.

If \( f(t) > g(t) > 0 \) and \( \int_0^\infty g(t) \, dt \) diverges then \( \int_0^\infty f(t) \, dt \) diverges.
Laplace transform: definition and existence

A special machine which changes one function into another.  
Input: \( f(t), \ t > 0 \).  
Output:

\[
F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st} \, dt
\]

Definition

If \( f \) is a piece-wise continuous function on each interval \([0, A]\) then

\[
F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st} \, dt = \lim_{A \to \infty} \int_0^A f(t)e^{-st} \, dt,
\]

if the limit exists.

Theorem

If \(|f(t)| < M e^{kt}\) (such \( f \) is called a function of exponential order), then \( \mathcal{L}\{f\}(s) \) exists for all \( s > k \).
First examples of the Laplace transform

**Example 1** $f(t) = 1$

$$F(s) = \lim_{A \to \infty} \int_{0}^{A} e^{-st} \, dt = \lim_{A \to \infty} \frac{e^{-st}}{s} \bigg|^{A}_{0}$$

$$= \lim_{A \to \infty} \left( -\frac{e^{-sA}}{s} + \frac{1}{s} \right) = \frac{1}{s}, \quad s > 0.$$

**Example 2** $f(t) = e^{at}$

$$F(s) = \int_{0}^{\infty} e^{at} e^{-st} \, dt = \int_{0}^{\infty} e^{-(s-a)t} \, dt = \frac{1}{s-a}, \quad s > a.$$
Further examples of the Laplace transform

The Laplace transforms of the following functions can be evaluated using the definition (and integration by parts sometimes)

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{s}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$t^n$</td>
<td>$\frac{n!}{s^{n+1}}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}$, $s &gt; a$</td>
</tr>
<tr>
<td>$\cos at$</td>
<td>$\frac{s}{s^2+a^2}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$\sin at$</td>
<td>$\frac{a}{s^2+a^2}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$u_c(t)$, $c &gt; 0$</td>
<td>$\frac{e^{-cs}}{s}$, $s &gt; 0$</td>
</tr>
</tbody>
</table>
Basic rules

- **Linearity**
  \[ \mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\} \]

- **First shift rule**
  \[ \mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s - a) \]

- **Second shift rule** (see the next lecture for details)
  \[ \mathcal{L}\{u_c(t)f(t - c)\}(s) = e^{-cs}\mathcal{L}\{f\}(s) \]

- **Derivatives** (see the next lecture)
  \[ \mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0), \quad \mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0), \]
  \[ \mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0) \]
Consider an initial value problem for a linear ODE with constant coefficients

\[ y^{(n)} + a_{n-1}y^{n-1} + \ldots + a_1y' + y = f \]

\[ y(0) = K_0, \ y'(0) = K_1, \ldots, y^{(n-1)}(0) = K_{n-1} \]

It can be solved by the following procedure:

- apply the Laplace transform to obtain an algebraic equation on \( Y = \mathcal{L}\{y\} \)
- solve this equation and find \( Y \)
- find \( y \) such that \( \mathcal{L}\{y\} = Y \) (inverse Laplace transform)

The last step could be non-trivial.
Example 3, Laplace transform by definition

Find the Laplace transform of the function

\[ f(t) = \begin{cases} 
1, & 0 \leq t \leq 1 \\
t, & 1 \leq t < \infty 
\end{cases} \]

We use the definition:

\[ F(s) = \int_0^\infty f(t)e^{-st} \, dt = \int_0^1 e^{-st} \, dt + \int_1^\infty te^{-st} \, dt = F_1(s) + F_2(s) \]

where \( F_1(s) = -\frac{e^{-st}}{s} \bigg|_0^1 = 1/s - e^{-s}/s \) and

\[ F_2(s) = \int_1^\infty t(-e^{-st}/s)' \, dt = -\frac{te^{-st}}{s} \bigg|_1^\infty + \int_1^\infty \frac{e^{-st}}{s} \, dt = \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2} \]

Then

\[ F(s) = \frac{1}{s} + \frac{e^{-s}}{s^2}, \quad s > 0 \]
Example 4, Laplace transform using the $s$-shift

Compute the Laplace transform of the function $e^{-3t} \cos 4t$ if we know that

$$\mathcal{L}(\cos bt)(s) = \frac{s}{s^2 + b^2}.$$ 

We have

$$\mathcal{L}(e^{-3t} \cos(4t))(s) = \mathcal{L}(\cos(4t))(s - (-3)) =$$

$$= \frac{s + 3}{(s + 3)^2 + 4^2} = \frac{s + 3}{s^2 + 6s + 25}.$$
Example 5, Inverse Laplace transform

Find the inverse Laplace transform of the function

\[ F(s) = \frac{2s}{s^2 - 2s - 3} \]

We will use the linearity property and table functions. We want to simplify the function \( F(s) \) using the partial fraction decomposition:

\[ F(s) = \frac{2s}{s^2 - 2s - 3} = \frac{a}{s - 3} + \frac{b}{s + 1} \]

Or \( 2s = a(s - 1) + b(s - 3) \) and \( a = 3/2, b = 1/2 \).

\[ F(s) = \frac{3}{2(s - 3)} - \frac{1}{2(s + 1)} \]

Then \( f(t) = \mathcal{L}^{-1}(F)(t) = \frac{3}{2} e^{3t} - \frac{1}{2} e^{t} \).