14.1 Line integral in the complex plane.
14.2 Cauchy's integral theorem

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Definition:Riemann sums

Let C be a smooth simple curve on the complex plain with end points z_0 and z_e. We consider a subdivision of this curve into small pieces by points $z_0, z_1, ..., z_n = z_e$ on the curve and on each part of the curve (z_j, z_{j+1}) we choose an additional point ζ_j. Let f be a continuous complex valued function on C. To a partition $\{z_0, ..., z_n\}$ with choosen points $\{\zeta_0, ...\zeta_{n-1}$ we assign the Riemann sum

$$S_n = f(\zeta_0)(z_1 - z_0) + f(\zeta_1)(z_2 - z_1) + ... + f(\zeta_{n-1})(z_n - z_{n-1})$$

Now,if $n \to \infty$ and the partitions are chosen such that $|z_{j+1} - z_j|$ tend to zero,then the sequence S_n has a limit, by the definition it is $\int_C f(z)dz$.

Example

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1. By the definition: take $z_k = \zeta_k = e^{ik\pi/2n}$, $k = 0, ..., n$, then

$$S_n = \sum_{k=0}^{n-1} e^{ik\pi/2n}(e^{i(k+1)\pi/2n} - e^{ik\pi/2n}) = \sum_{k=0}^{n-1} e^{ik\pi/n}(e^{i\pi/2n} - 1)$$

$$= \frac{e^{i\pi} - 1}{e^{i\pi/n} - 1}(e^{i\pi/2n} - 1) = \frac{e^{i\pi} - 1}{e^{i\pi/2n} + 1} \to -1$$

Properties of the integral

- Linearity

$$\int_C (c_1 f_1(z) + c_2 f_2(z))dz = c_1 \int_C f_1(z)dz + c_2 \int_C f_2(z)dz$$

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- If C consists of two pieces C_1 and C_2 then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

Estimate of the integral

Theorem

Suppose that f is a continuous function on a curve C. If $|f(z)| \leq M$ for each z on C and L is the length of C then

$$\left| \int_C f(z)dz \right| \leq LM$$

Parametrization of the curve

Another way of defining the integral is via parametrisation of the curve

Proposition

If a simple curve C is parametrized by a differentiable function $g : [a, b] \to \mathbb{C}$. Then

$$\int_C f(z)dz = \int_a^b f(g(t))g'(t)dt$$

Idea: compare the Riemann sum for the integral above:
$\sum_k f(g(s_k))g'(s_k)(t_{k+1} - t_k)$, $a = t_0 < s_0 < t_1 < s_1 < ... < t_n = b$,
with the Riemann sum for the integral of f over C which can be written as

$$\sum_k f(g(s_k))(g(t_{k+1}) - g(t_k)) \quad .$$

The difference between the sums goes to zero as $n \to \infty$.

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2 By a parametrization, $g(t) = e^{it}, 0 \le t \le \pi/2$, $g'(t) = ie^{it}$then

$$\int_C f(z)dz = \int_0^{\pi/2} e^{it} ie^{it} dt = i \int_0^{\pi/2} e^{2it} dt = \frac{1}{2} \; e^{2it}\big|_{t=0}^{t=\pi/2} = -1$$

Important example

Let $C = C_R = \{z : |z| = R\}$ be the circle of radius R centered at the origin, $f(z) = z^m$, $m = 0, \pm 1, \pm 2, \pm 3, \ldots$.

$$I_m(R) = \int_{C_R} z^m dz$$

Parametrization $g(t) = Re^{it}, 0 \leq t < 2\pi \Rightarrow$

$$I_m(R) = \int_0^{2\pi} R^m e^{itm}(iRe^{it})dt = iR^{m+1} \int_0^{2\pi} e^{i(m+1)t} dt$$
$$= \begin{cases} 0, & m \neq -1 \\ 2i\pi, & m = -1 \end{cases}$$

Remark. The answer does NOT depend upon R !

Change of variables

We will need integrals around circles centred at various points. Let $T_R = \{|z - z_0| = R\}$ and $f(z) = (z - z_0)^m$ then one can do a change of variable

$$\int_{|z-z_0|=R} (z - z_0)^m dz = \begin{cases} 0, & m \neq -1 \\ 2i\pi, & m = -1 \end{cases}$$

Remark. Pay attention to notation: $\int_{T_R}$ is written as $\int_{|z-z_0|=R}$

Reduction to real valued integrals

One more way to define the integral is reduction to real integrals over curves in plane.
If $f(z) = u(z) + iv(z)$, $dz = dx + idy$ then
$f(z)dz = (u + iv)(dx + idy) = (udx - vdy) + i(udy + vdx)$.

This can be taken as definition of the integral:

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy) = \int_C (udx-vdy)+i\int_C (udy+vdx)$$

Why do we need three definitions of integral

- Riemann sum definition: easy to prove the main facts;
- Parametrisation definition: easy to calculate
- 2D real definition: convenient to make statements about integration of analytic functions

Our main objective: Study the integrals of *analytic* functions along curves.

Simply connected domains

A domain D on the complex plain is *simply connected* if any simple closed curve in D is a boundary of a subdomain of D.

Example

1. Any circle is a simply connected domain.
2. A circular ring or a punched disc are not simply connected domains.

Intuitive definition of a simply connected domain: A domain without halls

Each domain bounded by a simple closed curve is simply connected

The most important theorem

Cauchy's integral theorem

Theorem

Let f be an analytic function in a simply connected domain D. If C is a simple closed curve in D then

$$\int_C f(z)dz = 0$$

Proof

Tool: Green formula from Calculus 2: (so we use real notation)

Let the curve C bound some region R in plane and $P(x, y)$, $Q(x, y)$ be continuous and differentialble in the closure of R. Then

$$\int_C (Pdx + Qdy) = \int\int_R \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dxdy$$

Now we use "real" definition of the integral and apply the Green formula using the fact that, since f is analytic we have the Cauchy-Riemann equations met:

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (udy + vdx)$$
$$= \int_R (-u_y - v_x)dxdy + i\int_R (u_x - v_y)dxdy = 0$$

Path independence

In a simply connected domain the integral $\int_C f(z)dz$ of an analytic function does not depend on the path C but only on its end points, we write also

$$\int_C f(z)dz = \int_{z_0}^{z_e} f(z)dz$$

Remark It IS important that domain is simply connected. Example: take $f(z) = 1/z$ in an annulus around the origin.

Anti-derivative

Theorem

Let f be an analytic function on a simply connected domain D. Then there is an analytic function F in D such that $F'(z) = f(z)$ for each z in D and

$$\int_C f(z)dz = F(z_e) - F(z_0)$$

where C is a simple curve with end points z_0 and z_e.

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To construct the anti-derivative we fix some point z_c in D and for each z in D define

$$F(z) = \int_{z_c}^{z} f(\zeta)d\zeta$$

One can check that F defined in this way is analytic and $F'(z) = f(z)$.