17.1 Geometry of analytic functions. Conformal mapping

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Functions of a complex variable are mappings from $\mathbb{C}$ (or a domain $D \subset \mathbb{C}$) to $\mathbb{C}$, $w = f(z)$, $z \in D$.

$D$ is the range of $f$, $f(D) = \{w = f(z), z \in D\}$ is the image of $D$ under action $f$.

Examples: images of curves, images of domains.
Example from last week: $f(z) = e^z$, images of the Cartesian coordinate net, images of strips, etc.

Our goal for today: get intuition (and some rigorous) knowledge about mapping by analytic functions.
Simplest examples

▷ **Shift:** Fix $a \in \mathbb{C}$. Let $w = f(z) = z + a$. This is a shift of the complex plane.

▷ **Rotation:** Fix $c \in \mathbb{C}$, $|c| = 1$. Let $w = f(z) = cz$. This is a rotation of the complex plane.

▷ **Scaling and rotation** Let $b \in \mathbb{C}$ be arbitrary, then $w = f(z) = cz$ is scaling by $|b|$ and rotation by $\text{Arg}(b)$.

▷ **Linear mapping:** $w = f(z) = a + bz$. This is a combination of shift, scaling and rotation of the complex plane.

Linear mappings preserve the angles between straight lines.

In order to have this fact for more general mappings we need a definition of angle between curves.
Angles between curves

Let $C_1$, $C_2$ be two curves which intersect at some point $z_0$.

The angle between $C_1$ and $C_2$ at $z_0$ is the angle between the tangent lines to $C_1$ and $C_2$ at this point.

A curve (in complex notation) is defined by:
$$C = \{ z(t) = x(t) + iy(t), t \in (a, b) \subset \mathbb{R} \}.$$

Examples: arcs, segments in $\mathbb{C}$, arbitrary curves.

Natural parametrization is by the arc length, such that $|\dot{z}(t)| = 1$.

Given $t_0 \in (a, b)$ and the corresponding point $z_0 = z(t_0) \in \mathbb{C}$, $\dot{z}(t_0)$ is directed along the tangent at $z_0$. 
A mapping is called conformal if it preserves angles between curves (including the direction).

Examples:
- Linear mappings are conformal
- The exponential mapping is conformal (so far we checked this just for horizontal and vertical lines)
- Reflections with respect lines are not conformal

Fact A mapping \( w = f(z) \) by an analytic function \( f \) is conformal at each point \( z \) where \( f'(z) \neq 0 \).

The inverse statement is also true: a conformal mapping with partial derivatives is an analytic function which derivative is not zero at each point.
Idea of the proof

If $f$ is analytic near $z_0$, then locally (i.e. when $z$ is close to $z_0$)

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(|z - z_0|),$$

this is the main part

The main part is a linear mapping!
Locally each analytic function is a shift and a scaling with rotation (where $f' \neq 0$).

To prove the inverse statement, one can express mathematically the angles between images of straight lines and deduce the Cauchy-Riemann equations.
Definition If $f(z)$ is analytic and if $f'(z_0) = 0$ then $z_0$ is called a critical point of $f$.

The mapping $w = f(z)$ defined by an analytic function is conformal except at critical points.

Q: What happens in the critical points?
Power function

\[ w = f(z) = z^\alpha, \; \alpha > 0. \]

This is an analytic function at \( z \neq 0 \) due to our definition of power function \( z^\alpha = e^{\alpha \ln z} \) and \( f'(z) = \alpha z^{\alpha - 1} \).

It also can be written as

\[ z = re^{i\phi} \Rightarrow w = r^\alpha e^{i\alpha\phi} \] - this mapping opens angles (if \( \alpha > 1 \)) or compress angles (if \( \alpha < 1 \)).

Spacial case: \( w = f(z) = z^\alpha, \; n > 0, \text{ integer}. \) Then \( f(z) \) is an analytic function at \( z = 0 \) as well, \( z = 0 \) is a critical point. Each angle with vertex at the origin is mapped into an angle which is \( n \)-times larger.
Further examples of conformal mappings

- Exponential function $f(z) = e^z$.
- Logarithmic function $f(z) = \ln z$

Definition Let $f$ maps a domain $S \subset \mathbb{C}$ one to one onto a domain $T \subset \mathbb{C}$, (i.e. $T = \{w = f(z), \, z \in S\}$ and for each $w \in T$ there is just one $z \in S$ such that $f(z) = w$) we can define the inverse mapping $f^{-1} : T \rightarrow S$:

$$z = f^{-1}(w) \text{ if } w = f(z)$$

Principle of inverse mapping: If $f$ is conformal then $f^{-1}$ is conformal as well.
Two more examples

Inversion: \( w = f(z) = \frac{1}{z} \)

Make pictures. Inversion maps the unit disc onto exterior of the unit disc.

Rule In order to trace the image of a domain we have to look at the image of its boundary.

Joukowski mapping: \( w = f(z) = z + \frac{1}{z} \).

▶ Derivative and the critical points
▶ Exterior (and interior) of the unit disc onto exterior of the segment
▶ Bigger discs onto ellipses
▶ Shifted discs onto ”Joukowski airfoil”