17.1 Geometry of analytic functions. Conformal mapping

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Geometrical viewpoint

Function of a complex variable: $w = f(z)$, $z \in D$.

D - *range* of f, $f(D) = \{w = f(z), z \in D\}$ - image of D under action f.

Examples: images of curves, images of domains.

Remind: $f(z) = e^z$, images of the cartesian coordinate net, images of strips etc.

Our goal for today: get intuition (and some rigorous) knowledge about mapping by analytic functions.

Simplest examples

- Shift: Fix $a \in \mathbb{C}$. Let $w = f(z) = z + a$. This is a shift of the complex plane
- Rotation/scaling: Fix $b \in \mathbb{C}$. Let $w = f(z) = bz$. This is a rotation and scaling of the complex plane
- Linear mapping: $w = f(z) = a + bz$. This is shift and rotation and scaling of the complex plane

Linear mappings preserve the angles between straight lines

In order to have this fact for more general mappings we need definition of angle between curves.

Angles between curves

Let C_1, C_2 be two curves which intersect at the point z_0.

Angle between C_1 and C_2 at z_0 is angle between their tangents at this point

Remind definition of curve (in complex notation):

$$C = \{z(t) = x(t) + iy(t), t \in (\alpha, \beta) \subset \mathbb{R}\}.$$

Examples: arcs, segments in $\mathbb{C}$, arbitrary curves.
Natural parametrisation: by the arc length.

Given $t_0 \in (\alpha, \beta)$ and the corresponding point $z_0 = z(t_0) \in \mathbb{C}$, $\dot{z}(t_0)$ is directed along the tangent at z_0.

Conformal mapping

Mapping is conformal if it preserves angles between curves.
Examples:
– Linear mapping
– Exponential mapping (so far we checked this just for horizontal and vertical lines)

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Fact The mapping $w = f(z)$ by an analytic function is conformal at each point z where $f'(z) \neq 0$

Actually the inverse statement is also true: if $f : S \to T$ is a mapping of two domains which is conformal at each point of S, then f is analytic and $f'(z) \neq 0$ for all $z \in S$.

Idea of the proof

If f is analytic near z_0, then locally (i.e. when z is close to z_0)

$$f(z) = \underbrace{f(z_0) + f'(z_0)(z - z_0)}_{\text{this is the main part}} + o(|z - z_0|),$$

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The main part is a linear mapping!
Locally each analytic function is a shift and a rotation (where $f' \neq 0$).

<u>Definition</u> $f(z)$ is analytic at z_0. If $f'(z_0) = 0$ then z_0 is a *critical point*

Q: What happens in the critical points ?

Power function

$$w = f(z) = z^\alpha,\ \alpha > 0.$$

This is an analytic function if $z \neq 0$ (remember our definition of power function $z^\alpha = e^{\alpha \ln z}$, also $f'(z) = \alpha z^{\alpha - 1}$)
It also can be written as

$z = re^{i\phi} \Rightarrow w = r^\alpha e^{i\alpha\phi}$ - this mapping opens angles (if $\alpha > 1$) or compress angles (if $\alpha < 1$).

Special case: $w = f(z) = z^n$, $n > 0$, integer. Then $f(z)$ is an analytic function at $z = 0$ as well, $z = 0$ is a critical point. Each angle with vertex at zero increases n times.

Question: How many times does $f(z) = z^n$ takes each value?

More examples

- Exponential function $f(z) = e^z$.
- Logarithmic function $f(z) = \ln z$

Definition Let f maps a domain $S \subset \mathbb{C}$ *one to one* onto a domain $T \subset \mathbb{C}$, (i.e. $T = \{w = f(z), z \in S\}$ and for each $w \in T$ there is just one $z \in S$ such that $f(z) = w$) we can define the inverse mapping $f^{-1} : T \to S$:

$$z = f^{-1}(w) \text{ if } w = f(z)$$

Principle of inverse mapping: If f is conformal then f^{-1} is conformal as well

More examples

Inversion: $w = f(z) = \frac{1}{z}$
Make pictures. Inversion maps the unit disc onto exterior of the unit disc.

Rule In order to trace the image of a domain we have to look at the image of its boundary.

Zhukovskii mapping: $w = f(z) = z + 1/z$.

- Derivative and the critical points
- Exterior (and interior) of the unit disc onto exterior of the segment
- Bigger discs onto ellipses
- Shifted discs onto "Zhukovskii airfoil"