17.1 Geometry of analytic functions. Conformal mapping

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Function of a complex variable: $w = f(z)$, $z \in D$.

$D$ - range of $f$, $f(D) = \{w = f(z), z \in D\}$ - image of $D$ under action $f$.
Examples: images of curves, images of domains.
Remind: $f(z) = e^z$, images of the cartesian coordinate net, images of strips etc.
Our goal for today: get intuition (and some rigorous) knowledge about mapping by analytic functions.
Simplest examples

- **Shift**: Fix $a \in \mathbb{C}$. Let $w = f(z) = z + a$. This is a shift of the complex plane.
- **Rotation/scaling**: Fix $b \in \mathbb{C}$. Let $w = f(z) = bz$. This is a rotation and scaling of the complex plane.
- **Linear mapping**: $w = f(z) = a + bz$. This is shift and rotation and scaling of the complex plane.

Linear mappings preserve the angles between straight lines.

In order to have this fact for more general mappings we need definition of angle between curves.
Angles between curves

Let $C_1$, $C_2$ be two curves which intersect at the point $z_0$.

*Angle between $C_1$ and $C_2$ at $z_0$ is angle between their tangents at this point*

Remind definition of curve (in complex notation):

$$C = \{z(t) = x(t) + iy(t), \ t \in (\alpha, \beta) \subset \mathbb{R}\}.$$ 

Examples: arcs, segments in $\mathbb{C}$, arbitrary curves.

Natural parametrisation: by the arc length.

Given $t_0 \in (\alpha, \beta)$ and the corresponding point $z_0 = z(t_0) \in \mathbb{C}$, $\dot{z}(t_0)$ is directed along the tangent at $z_0$. 
Mapping is *conformal* if it preserves angles between curves.

Examples:

– Linear mapping
– Exponential mapping (so far we checked this just for horizontal and vertical lines)
Mapping is **conformal** if it preserves angles between curves. Examples:
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- World map

Conformity is a local property.

**Fact**

The mapping \( w = f(z) \) by an analytic function is conformal at each point \( z \) where \( f'(z) \neq 0 \).

Actually the inverse statement is also true: if \( f : S \rightarrow T \) is a mapping of two domains which is conformal at each point of \( S \), then \( f \) is analytic and \( f'(z) \neq 0 \) for all \( z \in S \).
Conformal mapping

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Actually the inverse statement is also true: if $f : S \rightarrow T$ is a mapping of two domains which is conformal at each point of $S$, then $f$ is analytic and $f'(z) \neq 0$ for all $z \in S$. 
Idea of the proof

If $f$ is analytic near $z_0$, then locally (i.e. when $z$ is close to $z_0$)

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(|z - z_0|),$$

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The main part is a linear mapping!
Locally each analytic function is a shift and a rotation (where $f' \neq 0$).

**Definition** $f(z)$ is analytic at $z_0$. If $f'(z_0) = 0$ then $z_0$ is a critical point

**Q:** What happens in the critical points?
\[ w = f(z) = z^\alpha, \quad \alpha > 0. \]

This is an analytic function if \( z \neq 0 \) (remember our definition of power function \( z^\alpha = e^{\alpha \ln z} \), also \( f'(z) = \alpha z^{\alpha - 1} \))

It also can be written as

\[ z = re^{i\phi} \Rightarrow w = r^\alpha e^{i\alpha \phi} \] - this mapping opens angles (if \( \alpha > 1 \)) or compress angles (if \( \alpha < 1 \)).

Special case: \( w = f(z) = z^n, \quad n > 0, \text{ integer} \). Then \( f(z) \) is an analytic function at \( z = 0 \) as well, \( z = 0 \) is a critical point. Each angle with vertex at zero increases \( n \) times.

Question: How many times does \( f(z) = z^n \) takes each value?
More examples

- Exponential function $f(z) = e^z$.
- Logarithmic function $f(z) = \ln z$.

**Definition** Let $f$ maps a domain $S \subset \mathbb{C}$ *one to one* onto a domain $T \subset \mathbb{C}$, (i.e. $T = \{w = f(z), z \in S\}$ and for each $w \in T$ there is just one $z \in S$ such that $f(z) = w$) we can define the inverse mapping $f^{-1} : T \rightarrow S$:

$$z = f^{-1}(w) \text{ if } w = f(z)$$

**Principle of inverse mapping:** If $f$ is conformal then $f^{-1}$ is conformal as well.
More examples

Inversion: \( w = f(z) = \frac{1}{z} \)

Make pictures. Inversion maps the unit disc onto exterior of the unit disc.

Rule In order to trace the image of a domain we have to look at the image of its boundary.

Zhukovskii mapping: \( w = f(z) = z + \frac{1}{z} \).

- Derivative and the critical points
- Exterior (and interior) of the unit disc onto exterior of the segment
- Bigger discs onto ellipses
- Shifted discs onto ”Zhukovskii airfoil”