13.4-5 Cauchy-Riemann equations. Exponential function.

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Derivative (reminder)

$S \subset \mathbb{C}$ - domain,
$f : S \to \mathbb{C}$ analytic function; $f(z) = u(x,y) + iv(x,y)$

$$z_0 \in S: \; f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

The same limit no matter along which direction z approachhes z_0 !

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Q: How can this be expressed in terms of u and v - real and imaginary parts of f ?

Cauchy-Riemann conditions

Fact: It suffices that limits coincide along two directions only, then they coincide along all directions

$$\frac{f(z_0+\Delta x)-f(z_0)}{\Delta x} \to u_x + i v_x, \text{ as } \Delta x \to 0,$$

$$\frac{f(z_0+i\Delta y)-f(z_0)}{i\Delta y} \to -i u_y + v_y, \text{ as } \Delta y \to 0,$$

These expressions should be the same, so we compare the real and imaginary parts:

$$\boxed{u_x = v_y, \quad -u_y = v_x}$$

These are the Cauchy-Riemann conditions !

Cauchy-Riemann conditions (ctd)

$$\boxed{f'(z_0) \text{ exists } \Leftrightarrow u \text{ and } v \text{ satisfy the Cauchy Riemann conditions}}$$

Expressions for the derivative:

$$f'(z) = u_x + iv_x = -iu_y + v_y.$$

Examples

- $f(z) = z^2$. We know f' exists. Check the C-R conditions.
- $f(z) = e^z$. We verify the C-R conditions for this function, hence it is analytic.
- A function which can take real values only cannot be analytic unless is a constant

Observe our collection of analytic functions:

- polynomials
- rational functions (at the points where the denominator does not vanish)
- exponential function and its products on the above functions
- Composed functions, e^{z^2} for example

Real and imaginary parts of analytic function

Q: Which u and v can be real and imaginary parts of an analytic function?

First C-R condition: $u_x = v_y \Rightarrow u_{xx} = v_{xy}$.

Second C-R condition: $u_y = -v_x \Rightarrow u_{yy} = -v_{xy}$

Finally we obtain the Laplace equation!

$$\boxed{u_{xx} + u_{yy} = 0} \text{ similarly } \boxed{v_{xx} + v_{yy} = 0}$$

Such functions are called harmonic.

A couple of harmonic functions u, v which satisfy the C-R conditions, equivalently which are real and imaginary parts of some analytic functions are called harmonic conjugate

Remind notation for Laplacian:

$$\Delta u = \nabla^2 u = u_{xx} + u_{yy}$$

Conjugate harmonic functions

- Take real and imaginary parts of z^2, e^z, say.
- Let $u(x, y) = x^2 - y^2 + 3y$. Prove that u is a harmonic function and find its harmonic conjugate.
- (Exam 2004) $f(z) = y^3 + Bx^2y + iv(x, y)$ is analytic. Find B and then v such that $v(0, 0) = 0$.

The general procedure: given a harmonic u find its harmonic conjugate v.

Exponential function, properties

- $z = x + iy \Rightarrow \exp z = e^z = e^{x+iy} = e^x \cos y + ie^x \sin y$
- $|e^z| = e^x$, $\arg e^z = y$.
- e^z is analytic for all z, $(e^z)' = e^z$.
- algebraic property: $e^{z_1+z_2} = e^{z_1} e^{z_2}$
- periodicity: $e^{z+2i\pi} = e^z$

Exponential function, geometry

- Argument and absolute values
- e^{x+iy} runs along circles when x is fixed, and y varies.
- e^{x+iy} runs along rays when y is fixed, and x varies.
- Selected values: $e^{in\pi}$, $e^{i\pi/2}$, etc.
- Solution of the equation $e^z = w$
- Mapping of plane domains by the exponential function; strips and angles.