13.4-5 Cauchy-Riemann equations. Exponential function.

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Derivative (reminder)

\[ S \subset \mathbb{C} - \text{domain,} \]
\[ f : S \to \mathbb{C} \text{ analytic function; } f(z) = u(x, y) + iv(x, y) \]

\[ z_0 \in S : \quad f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \]

The same limit no matter along which direction \( z \) approaches \( z_0 \)!
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**Q:** How can this be expressed in terms of \( u \) and \( v \) - real and imaginary parts of \( f \)?
Cauchy-Riemann conditions

Fact: It suffices that limits coincide along two directions only, then they coincide along all directions

\[
\frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} \to u_x + iv_x, \text{ as } \Delta x \to 0, \\
\frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y} \to -iu_y + v_y, \text{ as } \Delta y \to 0,
\]

These expressions should be the same, so we compare the real and imaginary parts:

\[
u_x = v_y, \quad -u_y = v_x
\]

These are the Cauchy-Riemann conditions!
Cauchy-Riemann conditions (ctd)

\[ f'(z_0) \text{ exists} \iff u \text{ and } v \text{ satisfy the Cauchy Riemann conditions} \]

Expressions for the derivative:

\[ f'(z) = u_x + iv_x = -iu_y + v_y. \]
Examples

- $f(z) = z^2$. We know $f'$ exists. Check the C-R conditions.
- $f(z) = e^z$. We verify the C-R conditions for this function, hence it is analytic.
- A function which can take real values only cannot be analytic unless is a constant

Observe our collection of analytic functions:

- polynomials
- rational functions (at the points where the denominator does not vanish)
- exponential function and its products on the above functions
- Composed functions, $e^{z^2}$ for example
Q: Which $u$ and $v$ can be real and imaginary parts of an analytic function?

First C-R condition: $u_x = v_y \Rightarrow u_{xx} = v_{xy}$.

Second C-R condition: $u_y = -v_x \Rightarrow u_{yy} = -v_{xy}$

Finally we obtain the Laplace equation!

$$u_{xx} + u_{yy} = 0$$

Similarly

$$v_{xx} + v_{yy} = 0$$

Such functions are called **harmonic**.

A couple of harmonic functions $u, v$ which satisfy the C-R conditions, equivalently which are real and imaginary parts of some analytic functions are called **harmonic conjugate**

Remind notation for Laplacian:

$$\Delta u = \nabla^2 u = u_{xx} + u_{yy}$$
Conjugate harmonic functions

- Take real and imaginary parts of $z^2, e^z$, say.
- Let $u(x, y) = x^2 - y^2 + 3y$. Prove that $u$ is a harmonic function and find its harmonic conjugate.
- (Exam 2004) $f(z) = y^3 + Bx^2y + iv(x, y)$ is analytic. Find $B$ and then $v$ such that $v(0, 0) = 0$.

The general procedure: given a harmonic $u$ find its harmonic conjugate $v$. 
Exponential function, properties

- $z = x + iy \Rightarrow \exp z = e^z = e^{x+iy} = e^x \cos y + ie^x \sin y$
- $|e^z| = e^x$, $\arg e^z = y$.
- $e^z$ is analytic for all $z$, $(e^z)' = e^z$.
- Algebraic property: $e^{z_1 + z_2} = e^{z_1} e^{z_2}$
- Periodicity: $e^{z+2i\pi} = e^z$
Exponential function, geometry

- Argument and absolute values
- $e^{x+iy}$ runs along circles when $x$ is fixed, and $y$ varies.
- $e^{x+iy}$ runs along rays when $y$ is fixed, and $x$ varies.
- Selected values: $e^{i\pi}$, $e^{i\pi/2}$, etc.
- Solution of the equation $e^z = w$
- Mapping of plane domains by the exponential function; strips and angles.