13.3 Analytic functions (Analytiske funksjoner)

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Complex plane

Complex numbers considered as the points in the two-dimensional plane, we use new notation for the points:

$$z = (x, y) = x + iy = re^{i\theta},$$

$$-\infty < x, y < \infty, \ 0 \le r < \infty, -\pi < \theta \le 2\pi$$

Here $r = |z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$ is the absolute value of z (absoluttverdien), it is distance between the point $z = (x, y)$ and the origin $(0, 0)$.

$x = \Re(z)$ is the real part (reelldelen) and $y = \Im(z)$ is the imaginary part (imaginærdelen) of z, the complex conjugate of z (den kompleks konjugerte til z) is $\bar{z} = x - iy$.

Basic sets

Circular domains We fix a complex number z_0. Then

- $\{z : |z - z_0| = R\}$ is a circle of radius R centered at z_0,
- $\{z : |z - z_0| < R\}$ is an open disk of radius R and center z_0,
- $\{z : |z - z_0| \leq R\}$ is a closed disk of radius R and center z_0,
- $\{z : r < |z - z_0| < R\}$ is an open circular ring (annulus) of radii $r < R$ and center z_0.

Half-Planes Let $z = x + iy$

- The upper half-plane is the set of points with $y > 0$ and the lower half-plane is the set where $y < 0$.
- The right half-plane is the set where $x > 0$, the left half-plane is where $x < 0$.

Point sets (Punktmengder) (vocabulary)

Let S be a set of points on the complex plane.

- S is called **open** (åpen) if for each point $z \in S$ there is a disk centered at z which is contained in S
- S is called **linearly connected** (sammenhengende) if for any two points z_1 and z_2 in S there is a continuous curve γ with end-points z_1 and z_2 which is contained in S (a continuous curve is a continuous mapping $\gamma : [0, 1] \to \mathbb{C}$)
- S is called a **domain** (omegn) if S is open and linearly connected.

Continuous functions

Let D be a domain in $\mathbb{C}$, consider a function $f : D \to \mathbb{C}$. It is called continuous at point z_0 if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $|z - z_0| < \delta$ then $z \in D$ and $|f(z) - f(z_0)| < \epsilon$.

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Let $f(z) = u(z) + iv(z)$, where $u, v : D \to \mathbb{R}$. Then f is continuous at $z_0 = (x_0, y_0)$ if and only if u and v are continuous at this point.

Examples of continuous functions

- $f(z) = |z|$ is continuous everywhere,
- $f(z) = \mathrm{Arg(z)}$ is discontinuous at points $z = x + 0i, x \leq 0$,
- $f(z) = \Re(z) = x, f(z) = \Im(z) = y, f(z) = z, f(z) = \bar{z}$ are continuous everywhere,
- $f(z) = e^z = e^x e^{iy}$ is continuous everywhere.

Combinations of continuous functions (sums, differences, products, compositions) are continuous. Everything is as for real-valued functions of two variables.

Derivative

Let $f : D \to \mathbb{C}$ be a continuous function. We say that f is differentiable at some point $z_0 \in D$ if the following limit exists

$$f'(z_0) = \lim_{w \to 0} \frac{f(z_0 + w) - f(z_0)}{w}$$

Remember that w here is a complex number!

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However this is VERY different from the derivatives in Calculus 1.

Good old examples

- $f(z) = C$, then $\frac{f(z)-f(z_0)}{z-z_0} = 0$, the constant function is differentiable with $f'(z_0) = 0$.
- $f(z) = z$, then $\frac{f(z)-f(z_0)}{z-z_0} = 1$, the function is differentiable and $f'(z_0) = 1$,
- $f(z) = z^2$ then $\frac{f(z_0+w)-f(z_0)}{w} = 2z_0 + w \to 2z_0$ as $w \to 0$, $f'(z_0) = 2z_0$
- $f(z) = c_k z^k + c_{k-1} z^{k-1} + ... + c_1 z + c_0$ is a polynomial, then f is differentiable at each point and $f'(z) = k c_k z^{k-1} + (k-1) c_{k-1} z^{k-2} + ... + c_1$

Bad new examples

- $f(z) = \Re(z) = x$, then $\frac{f(z+w)-f(z)}{w} = \frac{\Re(w)}{w}$ has no limit as $w \to 0$! This function is not differentiable,
- $f(z) = \bar{z}$, then $\frac{f(z+w)-f(z)}{w} = \frac{\bar{w}}{w}$ has no limit as $w \to 0$, not differentiable
- $f(z) = |z|^2$, then $f(z) = z\bar{z}$ and $\frac{f(z+w)-f(z)}{w} = \frac{z\bar{w}+\bar{z}w}{w} = \bar{z} + z\frac{\bar{w}}{w}$, the limit exists only when $z = 0$, $f'(0) = 0$ but f is not differentiable at $z \neq 0$.
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Q: What should we demand from $f = u + iv$ in order to grant existence of f'?