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Wave equation

We model small vibrations of an elastic homogeneous string, assume that the string performs small motion in vertical direction only.

Physical assumptions:

- The string is homogeneous and elastic.
- The gravitational force can be neglected.
- Each part of the string moves only vertically.

We are looking for a function $u(x,t)$ that describes the motion. The equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{T}{\rho}$$

where T is the tension of the string and ρ is the density. The equation is called one-dimensional wave-equation.

Separation of variables

As before, we are looking at solutions of the form $u(x,t) = F(x)G(t)$. And we get coupled equations for F and G:

$$\frac{F''}{F} = \frac{G''}{c^2 G} = k$$

Suppose that two ends of the string are fixed, $u(0,t) = u(L,t) = 0$, then we want to find solutions with $F(0) = F(L) = 0$. As earlier we obtain

$$F_n(x) = \sin\frac{n\pi x}{L}, \ n \text{ is integer}, \quad k_n = -\left(\frac{n\pi}{L}\right)^2 = -p_n^2$$

Then for G we obtain

$$G_n(t) = A_n \cos cp_n t + B_n \sin cp_n t$$

When $n = 1$ we obtain the fundamental mode of the string, $\omega_1 = cp_1/2\pi = \sqrt{T/\rho}/2L$ is the corresponding fundamental frequency.

Initial conditions

To describe the motion of the string we need to know its initial position $f(x) = u(x, 0)$ and initial velocity $g(x) = u_t(x, 0)$.
We look for a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos cp_n t + B_n \sin cp_n t) \sin p_n x$$

The initial conditions give

$$f(x) = \sum_{n=1}^{\infty} A_n \sin p_n x, \quad g(x) = \sum_{n=1}^{\infty} cp_n B_n \sin p_n x$$

D'Alembert solution

Consider the wave equation $u_{tt} = c^2 u_{xx}$ and introduce new variables $x = x + ct$ and $w = x - ct$. Then

$$u_x = u_v v_x + u_w w_x = u_v + u_w, \quad u_{xx} = u_{vv} + 2u_{vw} + u_{ww}$$

$$u_y = u_v v_y + u_w w_y = c(u_v - u_w), \quad u_{yy} = c^2(u_{vv} - 2u_{vw} + u_{ww}$$

In the new variables the equation is $u_{vw} = 0$!

Integrating first with respect to v and then with respect to w we get

$$u(x, y) = \phi(v) + \psi(w) = \phi(x + ct) + \psi(x - ct),$$

where ϕ and ψ are arbitrary functions.

Initial Conditions

To determine ϕ and ψ we should specify the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Then $\phi + \psi = f$ and $c(\phi' + \psi') = g$. Then

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)ds + C$$

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)ds - C$$

Then

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

Second order linear PDEs of two variables

The principle part (terms that contain only highest order derivatives) of a second order PDE of two variables has the following form

$$L(u) = Au_{xx} + 2Bu_{xy} + Cu_{yy}$$

We consider constant coefficient case (A, B, C are constants). We distinguish three types of equations

Type	Condition	Example	Normal form
Hyperbolic	$AC - B^2 < 0$	Wave: $u_{tt} - c^2 u_{xx}$	u_{vw}
Parabolic	$AC - B^2 = 0$	Heat: u_{xx}	u_{ww}
Elliptic	$AC - B^2 > 0$	Laplace: $u_{xx} + u_{yy}$	$u_{vv} + u_{ww}$

Normal form

To find the new variables v, w in which the equation has normal form we take $v = y + \lambda_1 x,\ \ w = y + \lambda_2 x$, where

- If the equation is hyperbolic, then $\lambda_{1,2}$ are the roots of $A\lambda^2 + 2B\lambda + C = 0$.
- If the equation is parabolic, then λ_1 is the root of $A\lambda^2 + 2B\lambda + C = 0$ and λ_2 is arbitrary.
- If the equation is elliptic, then $\lambda_{1,2}$ are the roots of the equation $A\lambda^2 + 2B\lambda + (2B^2/A - C) = 0$.

Heat equation on the line

Now we consider the equation $u_t = c^2 u_{xx}$ for $-\infty < x < \infty$and $t \geq 0$ with initial condition $u(x,0) = f(x)$. We assume that $f(x) \to 0$ as $x \to \pm\infty$ fast enough. Let us further suppose that $u(x,t)$ decreases fast when $t$is fixed and $x \to \pm\infty$ such that for each fixed t we can compute the Fourier transform$\hat{u}(w,t)$ of $u(x,t)$ with respect to x.Then

$$\hat{u}_t(w,t) = -c^2 w^2 \hat{u}(w,t) \quad \Rightarrow \quad \hat{u}(w,t) = C(w)e^{-c^2 w^2 t}$$

We find $C(w)$from the initial condition,
$C(w) = \hat{u}(w,0) = \hat{f}(w)$and using the inversion formula for the Fourier transform we get

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{-c^2 w^2 t} e^{iwx} dw =$$

$$\frac{1}{2\pi} \int \int f(y) e^{iw(x-y)} e^{-c^2 w^2 t} dy dw = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(y) e^{-\frac{(x-y)^2}{4c^2 t}} dy$$

Heat kernel

The last formula has the following form

$$u(x,t) = f(y) * k(y,t)$$

where

$$k(y,t) = \frac{1}{2c\sqrt{\pi t}} e^{-y^2/(4c^2 t)}$$

it is called the heat kernel (in dimension one).