12.3-4 Wave equation, D’Alembert solutions, classification
// 12.7 Heat equation on the line

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We model small vibrations of an elastic homogeneous string, assume that the string performs small motion in vertical direction only.

Physical assumptions:

- The string is homogeneous and elastic.
- The gravitational force can be neglected.
- Each part of the string moves only vertically.

We are looking for a function \( u(x, t) \) that describes the motion. The equation is

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{T}{\rho}
\]

where \( T \) is the tension of the string and \( \rho \) is the density. The equation is called one-dimensional wave-equation.
Separation of variables

As before, we are looking at solutions of the form 
\( u(x, t) = F(x)G(t) \). And we get coupled equations for \( F \) and \( G \):

\[
\frac{F''}{F} = \frac{G''}{c^2 G} = k
\]

Suppose that two ends of the string are fixed, 
\( u(0, t) = u(L, t) = 0 \), then we want to find solutions with 
\( F(0) = F(L) = 0 \). As earlier we obtain

\[
F_n(x) = \sin \left( \frac{n\pi x}{L} \right), \quad n \text{ is integer,} \quad k_n = -\left( \frac{n\pi}{L} \right)^2 = -\rho_n^2
\]

Then for \( G \) we obtain

\[
G_n(t) = A_n \cos cp_n t + B_n \sin cp_n t
\]

When \( n = 1 \) we obtain the fundamental mode of the string, 
\( \omega_1 = cp_1/2\pi = \sqrt{T/\rho}/2L \) is the corresponding fundamental frequency.

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To describe the motion of the string we need to know its initial position \( f(x) = u(x, 0) \) and initial velocity \( g(x) = u_t(x, 0) \).

We look for a solution of the form

\[
    u(x, t) = \sum_{n=1}^{\infty} (A_n \cos cp_n t + B_n \sin cp_n t) \sin p_n x
\]

The initial conditions give

\[
    f(x) = \sum_{n=1}^{\infty} A_n \sin p_n x, \quad g(x) = \sum_{n=1}^{\infty} cp_n B_n \sin p_n x
\]
Consider the wave equation $u_{tt} = c^2 u_{xx}$ and introduce new variables $x = x + ct$ and $w = x - ct$. Then

$$u_x = u_v v_x + u_w w_x = u_v + u_w, \quad u_{xx} = u_{vv} + 2u_{vw} + u_{ww}$$

$$u_y = u_v v_y + u_w w_y = c(u_v - u_w), \quad u_{yy} = c^2(u_{vv} - 2u_{vw} + u_{ww})$$

In the new variables the equation is $u_{vw} = 0!$

Integrating first with respect to $v$ and then with respect to $w$ we get

$$u(x, y) = \phi(v) + \psi(w) = \phi(x + ct) + \psi(x - ct),$$

where $\phi$ and $\psi$ are arbitrary functions.
To determine $\phi$ and $\psi$ we should specify the initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$. Then $\phi + \psi = f$ and $c(\phi' + \psi') = g$. Then

$$\phi(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_{x_0}^{x} g(s) ds + C$$

$$\psi(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_{x_0}^{x} g(s) ds - C$$

Then

$$u(x, t) = \frac{1}{2} (f(x + ct) + f(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$$
Second order linear PDEs of two variables

The principle part (terms that contain only highest order derivatives) of a second order PDE of two variables has the following form

\[ L(u) = Au_{xx} + 2Bu_{xy} + Cu_{yy} \]

We consider constant coefficient case \((A, B, C \text{ are constants})\). We distinguish three types of equations

<table>
<thead>
<tr>
<th>Type</th>
<th>Condition</th>
<th>Example</th>
<th>Normal form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hyperbolic</td>
<td>(AC - B^2 &lt; 0)</td>
<td>Wave: (u_{tt} - c^2 u_{xx})</td>
<td>(u_{vw})</td>
</tr>
<tr>
<td>Parabolic</td>
<td>(AC - B^2 = 0)</td>
<td>Heat: (u_{xx})</td>
<td>(u_{ww})</td>
</tr>
<tr>
<td>Elliptic</td>
<td>(AC - B^2 &gt; 0)</td>
<td>Laplace: (u_{xx} + u_{yy})</td>
<td>(u_{vv} + u_{ww})</td>
</tr>
</tbody>
</table>
To find the new variables $v, w$ in which the equation has normal form we take $v = y + \lambda_1 x, \ w = y + \lambda_2 x$, where

- If the equation is hyperbolic, then $\lambda_{1,2}$ are the roots of $A\lambda^2 + 2B\lambda + C = 0$.
- If the equation is parabolic, then $\lambda_1$ is the root of $A\lambda^2 + 2B\lambda + C = 0$ and $\lambda_2$ is arbitrary.
- If the equation is elliptic, then $\lambda_{1,2}$ are the roots of the equation $A\lambda^2 + 2B\lambda + (2B^2/A - C) = 0$. 
Heat equation on the line

Now we consider the equation $u_t = c^2 u_{xx}$ for $-\infty < x < \infty$ and $t \geq 0$ with initial condition $u(x, 0) = f(x)$. We assume that $f(x) \to 0$ as $x \to \pm\infty$ fast enough. Let us further suppose that $u(x, t)$ decreases fast when $t$ is fixed and $x \to \pm\infty$ such that for each fixed $t$ we can compute the Fourier transform $\hat{u}(w, t)$ of $u(x, t)$ with respect to $x$. Then

$$\hat{u}_t (w, t) = -c^2 w^2 \hat{u}(w, t) \Rightarrow \hat{u}(w, t) = C(w) e^{-c^2 w^2 t}$$

We find $C(w)$ from the initial condition, $C(w) = \hat{u}(w, 0) = \hat{f}(w)$ and using the inversion formula for the Fourier transform we get

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{-c^2 w^2 t} e^{iwx} dw =$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int f(y) e^{i w (x - y)} e^{-c^2 w^2 t} dy dw = \frac{1}{2c\sqrt{\pi} t} \int_{-\infty}^{\infty} f(y) e^{-\frac{(x-y)^2}{4c^2 t}} dy$$
The last formula has the following form

\[ u(x, t) = f(y) \ast k(y, t) \]

where

\[ k(y, t) = \frac{1}{2c\sqrt{\pi t}} e^{-y^2/(4c^2 t)} \]

it is called the heat kernel (in dimension one).