Equations on the line: 12.4 Wave equation, D'Alembert solutions; 12.7 Heat equation

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Wave equation

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Physical assumptions:

- The string is homogeneous and elastic.
- The gravitational force can be neglected.
- Each part of the string moves only vertically.

We are looking for a function $u(x, t)$ that describes the motion. The equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{T}{\rho}$$

where T is the tension of the string and ρ is the density. The equation is called one-dimensional wave-equation.

D'Alembert solution

Consider the wave equation $u_{tt} = c^2 u_{xx}$ and introduce new variables $v = x + ct$ and $w = x - ct$. Then

$$u_x = u_v v_x + u_w w_x = u_v + u_w, \quad u_{xx} = u_{vv} + 2u_{vw} + u_{ww}$$

$$u_t = u_v v_t + u_w w_t = c(u_v - u_w), \quad u_{tt} = c^2(u_{vv} - 2u_{vw} + u_{ww})$$

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Integrating first with respect to v and then with respect to w we get

$$u(x,t) = \phi(v) + \psi(w) = \phi(x + ct) + \psi(x - ct),$$

where ϕ and ψ are arbitrary functions.

Heat equation on the line

$$\text{Equation:} \quad u_t = c^2 u_{xx}, \ -\infty < x < \infty, \ t \geq 0; \qquad (*)$$

$$\text{Initial condition:} \quad u(x,0) = f(x).$$

$$\text{Assumption:} \quad f(x) \to 0, \ u(x,t) \to 0 \text{ as } x \to \pm\infty \text{ fast enough}$$

We can apply the Fourier transform in x ! Let

$$\hat{u}(w,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-iwx} dx.$$

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Then $(*) \Rightarrow$

$$\hat{u}_t(w,t) = -c^2 w^2 \hat{u}(w,t) \quad \Rightarrow \quad \hat{u}(w,t) = C(w) e^{-c^2 w^2 t}$$

We find $C(w)$ from the initial condition.
$t = 0 \Rightarrow C(w) = \hat{u}(w,0) = \hat{f}(w)$ and finally

$$\hat{u}(w,t) = e^{-c^2 w^2 t} \hat{f}(w)$$

Inverting Fourier transform for the heat equation

Remind the convolution theorem:

$$\mathcal{F}(f * g) = \sqrt{2\pi}\hat{f}\hat{g};$$

$$\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}}e^{-\frac{w^2}{4a}}.$$

$\Rightarrow$

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(y)e^{-\frac{(x-y)^2}{4c^2 t}}\,dy$$

Heat kernel

The last formula has the following form

$$u(x,t) = f(y) * k(y,t)$$

where

$$k(y,t) = \frac{1}{2c\sqrt{\pi t}} e^{-y^2/(4c^2 t)}$$

it is called the heat kernel (in dimension one).