12.6 Heat equation, 12.2-3 Wave equation

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September 29, 2016
The heat equation in higher dimensions (two or three) is

\[
\frac{\partial u}{\partial t} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad \frac{\partial u}{\partial t} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)
\]

We study steady solutions (that does-not depend on time \( t \)). Then the equation becomes

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}
\]

Solutions are called harmonic functions (\( \Delta u = 0 \)), \( \Delta u = u_{xx} + u_{yy} + u_{zz} \), \( \Delta \) is called the Laplace operator.
Separation of variables for 2D Laplace

We are looking for solutions of \( u_{xx} + u_{yy} = 0 \) on \( 0 \leq x \leq a, 0 \leq y \leq b \) of the form \( u(x, y) = F(x)G(y) \) then the equation becomes

\[
\frac{G''}{G} = -\frac{F''}{F} = \text{const}
\]

Boundary condition for Laplace equation:

- Dirichlet, function is known on the boundary, \( u(0, y), u(a, y), u(x, 0), u(x, b) \) are given;
- Neumann, the normal derivative is known in the boundary \( u_x(0, y), u_x((a, y), u_y(x, 0), u_y(x, b) \) are given
- Robin (mixed), function on a part of the boundary and normal derivative on the other part.
We assume that the boundary conditions are:

\[ u(0, y) = c_1, \quad u(a, y) = c_2, \]
\[ u(x, 0) = u(x, b) = c_1 + (c_2 - c_1)(x/a)^2 \]

1. Simplify the boundary condition (for example subtract some simple solution to get \( u(0, y) = u(a, y = 0) \))
2. Use separation of variables with the boundary condition in \( x \)
3. Combine solutions from separated variables to satisfy the other boundary condition
We can look at the function

$$v(x, y) = u(x, y) - c_1 - (c_2 - c_1)(x/a)$$

it is also harmonic (since $k_1 + k_2x$ is harmonic) and it satisfies

$$v(0, y) = v(a, y) = 0$$

The other boundary condition for $v$ will be

$$v(x, 0) = v(x, b) = (c_2 - c_1)((x/a)^2 - x/a) = cx(x - a)$$
Separation of variables

Now we are looking for solutions of the form $F(x)G(y)$ that satisfy $F(0) = F(a) = 0$. We have

$$\frac{G''}{G} = -\frac{F''}{F} = k_n$$

Then $F_n(x) = \sin(n\pi x/a)$, $k_n = -n^2\pi^2/a^2 = -p_n^2$ are solutions and the corresponding $G_n$ are

$$G_n(y) = A_n \cosh p_n y + B_n \sinh p_n y$$

We get $v_n(x, y) = F_n(x)g_n(y)$ are solutions.
Combination of product solutions

Then

\[ v(x, y) = \sum_{n=1}^{\infty} v_n(x, y) = \sum_{n=1}^{\infty} \left( A_n \cosh p_n y + B_n \sinh p_n y \right) \sin p_n x \]

is also a solution. We want to take a combination which satisfies additional boundary conditions. Boundary conditions on two other sides of the rectangle are

\[ v(x, 0) = v(x, b) = cx(x - a) \]

We set \( y = 0 \) and \( y = b \),

\[ cx(x - a) = \sum_{n=1}^{\infty} A_n \sin p_n x = \sum_{n=1}^{\infty} \left( A_n \cosh p_n b + B_n \sinh p_n b \right) \sin p_n x \]

We first find \( A_n \) from the first equality (take the Fourier series of the function \( x(x - a) \)) and then find \( B_n \) from the second equality:

\[ B_n \sinh p_n b = A_n \left( 1 - \cosh p_n b \right) \]
Fourier series

Let \( f(x) = cx(x - a) \) for \( 0 < x < a \). We extend \( f \) to an odd \( 2a \)-periodic function and compute the Fourier coefficients:

\[
b_n = \frac{2c}{a} \int_{0}^{a} x(x - a) \sin \frac{n\pi x}{a} \, dx = \begin{cases} 
-\frac{8ca^3}{n^3\pi^3}, & \text{if } n \text{ is odd} \\
0, & \text{if } n \text{ is even}
\end{cases}
\]

We take \( A_n = b_n \) and \( B_n = -b_n \coth \frac{p_n b}{2} \). Then the solution is

\[
v(x, y) = -\frac{8ca^3}{\pi^3} \sum_{k=1}^{\infty} \frac{\sin w_k x}{(2k - 1)^3} \left( \cosh w_k y - \coth \frac{bw_k}{2} \sinh w_k y \right),
\]

where \( w_k = p_{2k-1} = \frac{(2k-1)\pi}{a} \). Finally, the temperature distribution \( u(x, y) \) that we were looking for is given by

\[
u(x, y) = c_1 + \frac{(c_2 - c_1)x}{a} + v(x, y)
\]
Wave equation

We model small vibrations of an elastic homogeneous string, assume that the string performs small motion in vertical direction only.

Physical assumptions:

- The string is homogeneous and elastic.
- The gravitational force can be neglected.
- Each part of the string moves only vertically.

We are looking for a function $u(x, t)$ that describes the motion. The equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{T}{\rho}$$

where $T$ is the tension of the string and $\rho$ is the density. The equation is called one-dimensional wave-equation.
As before, we are looking at solutions of the form $u(x, t) = F(x)G(t)$. And we get coupled equations for $F$ and $G$:

$$\frac{F''}{F} = \frac{G''}{c^2G} = k$$

Suppose that two ends of the string are fixed, $u(0, t) = u(L, t) = 0$, then we want to find solutions with $F(0) = F(L) = 0$. As earlier we obtain

$$F_n(x) = \sin \frac{n\pi x}{L}, \ n \text{ is integer}, \ k_n = -\left(\frac{n\pi}{L}\right)^2 = -p_n^2$$

Then for $G$ we obtain

$$G_n(t) = A_n \cos cp_n t + B_n \sin cp_n t$$

When $n = 1$ we obtain the fundamental mode of the string.
To describe the motion of the string we need to know its initial position \( f(x) = u(x, 0) \) and initial velocity \( g(x) = u_t(x, 0) \).

We look for a solution of the form

\[
u(x, t) = \sum_{n=1}^{\infty} (A_n \cos cp_n t + B_n \sin cp_n t) \sin cp_n x
\]

The initial conditions give

\[
f(x) = \sum_{n=1}^{\infty} A_n \sin cp_n x, \quad g(x) = \sum_{n=1}^{\infty} cp_n B_n \sin cp_n x
\]
Method works for other equations

Example

a) Find all functions of the form $u(x, t) = F(x)G(t)$ that satisfy the differential equation

$$u_t + (1 + t^2)(2u - u_{xx}) = 0, \quad 0 \leq x \leq \pi/2, \quad t > 0$$

and the boundary conditions $u(0, t) = 0$, $u_x(\pi/2, t) = 0$.

b) Find the solution of the above equation that also satisfies the initial condition $u(x, 0) = \sin 3x + \sin 17x$. 
Solution (a), part 1

Separating variables in the equation, we obtain

\[
\frac{G'}{(1 + t^2)G} + 2 = \frac{F''}{F} = \text{const}
\]

The boundary condition implies \( F(0) = 0 \) and \( F'(\pi/2) = 0 \). We have from the equation \( F'' = cF \) then

- If \( c = 0 \) we get \( F(x) = ax + b \) and boundary conditions give \( a = b = 0 \) then \( F = 0 \). It gives only the trivial solution \( u = 0 \).
- If \( c = p^2 > 0 \) then \( F(x) = ae^{px} + be^{-px} \) (or one can use \( F(x) = c_1 \cosh px + c_2 \sinh px \)). The boundary conditions give \( a + b = 0 \) and \( pae^{p\pi/2} - pbe^{-p\pi/2} = 0 \) then \( a = b = 0 \), only the trivial solution.
- If \( c = -p^2 < 0 \) then \( F(x) = a \cos px + b \sin px \) and the boundary conditions give \( a = 0 \), \( pb \cos p\pi/2 = 0 \). We get a non-trivial solution when \( p = (2k + 1) \) with integer \( k \).

\[
F_k(x) = \sin(2k + 1)x
\]
Then the equation for $G$ is

$$\frac{G'_k}{G_k} = -((2k + 1)^2 + 2)(1 + t^2) = -c_k(1 + t^2)$$

Integrating, we get

$$\ln G_k(t) = -c_k(t + t^3/3) + C, \quad G_k(t) = C_k e^{-c_k(t + t^3/3)}$$

All solutions of the form $F(x)G(t)$ are given by

$$u_k(x, t) = F_k(x)G_k(t) = C_k \sin(2k + 1)x e^{-(4k^2+4k+3)(t+t^3/3)}$$
Solution (b)

If we have $u(x, t) = \sum_{k=0}^{\infty} u_k(x, t)$ then the initial condition is

$$u(x, 0) = \sum_{k=0}^{\infty} C_k \sin(2k + 1)x$$

We have $u(x, 0) = \sin 3x + \sin 17x$ then $C_1 = C_8 = 1$ and all other coefficients are zeros, we get

$$u(x, t) = \sin 3x e^{-11t} + \sin 17x e^{-291t}$$