12.1 Basics on PDE, 12.5-6Heat equation

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Definition, examples

P(artial) D(ifferential) E(quation) - equation on a function of several variables, which contains its partial derivatives.

Examples $u = u(x, t)$

- $$-\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x^2} + u = 0$$
- $$-\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x^2} + u = t$$
- $$-\frac{\partial^2 u}{\partial t^2} + (1 + \cos t)\frac{\partial^2 u}{\partial x^2} + u = t$$
- $$-\frac{\partial^2 u}{\partial t^2} + (1 + \cos u)\frac{\partial^3 u}{\partial x^3} + u = 0$$

Simplest classification

The highest derivative involved determines the order of the equation. We distinguish between linear and non-linear equations.

- $$-\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x^2} + u = 0$$
  order 2, linear, homogeneous, with constant coefficients.
- $$-\frac{\partial u}{\partial t} + (1 + \cos t)\frac{\partial^2 u}{\partial x^2} + u = t$$
  order 2, linear,.... with non-constant coefficients
- $$-\frac{\partial^2 u}{\partial t^2} + (1 + \cos u)\frac{\partial^3 u}{\partial x^3} + u = 0$$
  order 3, nonlinear

PDE are models for real processes:

- $u = u(x,t)$, $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$ 1D wave equation;
- $u = u(x,t)$, $\frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2}$ 1D heat equation;
- $u = u(x,y)$, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ 2D Laplace equation
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List of questions:

- Does the PDE have any solution?
- What kind of additional data should we specify in order to solve PDE and described a given process
- Is the solution unique

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And this is only the beginning (stability, singularity, etc.)

Linear equations

$$Lu = f \quad (*)$$

where L is a linear expression. Examples - all equations from the list above.

Homogeneous equation $\Leftrightarrow$ $f = 0$

Superposition principle:

if $u_1, u_2, \ \dots \ , u_n$ are solutions to a linear homogeneous equation, then $c_1 u_1 + c_2 u_2 + \ldots + c_n u_n$ also is a solution to this equation.

Superposition principle (non-homogeneous equation):

If u_1 and u_2 are the solutions of (*), then $u_1 - u_2$ solves the corresponding homogeneous equation $Lu = 0$

Heat equation

Physical assumptions:

- The amount heat per unit of mass is proportional to temperature
- The law of heat transfer: The rate of heat transfer is proportional to temperature gradient.
- Conservation of energy

The equation we get is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

and in higher dimensions

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Initial/boundary value problem for the 1D heat equation

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Boundary conditions (at the endpoints of the rod):

Fixed temperature: $u(x,0) = u_1$, $u(x,L) = u_2$,

or

Insulated ends: $u_x(x,0) = 0$, $u_x(x,L) = 0$

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Initial conditions (initial distribution of temperature):

$$u(x,0) = f(x), \; 0 < x < L.$$

Initial/boundary value problem for the 1D heat equation

We start with the simplest problem

$$\frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2}, \; 0 < x < L, \; t > 0,$$
$$u(0,t) = 0, \; u(L,t) = 0, \; t > 0,$$
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Idea: find many simple (elementary) solutions which meet the first two equations, and then using linearity combine them in order to meet the initial condition.

Elementary solutions (Separation of variables)

Assume that there is a solution of the 1D heat equation of the form $u(x,t) = X(x)T(t)$ then the equation becomes

$$\frac{T'}{c^2 T} = \frac{X''}{X} = \kappa$$

We obtained a couple of *ordinary* differential equations:

$$T'(t) - c^2\kappa T(t) = 0, t > 0; \quad X''(x) = \kappa X(x),\ 0 < x < L.$$

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But how to chose κ?

Boundary condition !

The last equation:

$$X''(x) = \kappa X(x),\ 0 < x < L \text{ and } X(0) = 0,\ X(L) = 0$$

This yields κ is negative, $\kappa = -p^2$, say, and

$$p = p_n = \frac{n\pi}{L},\ n = 1, 2,\ \ldots,\ X_n(x) = A_n \sin p_n x = A_n \sin(\frac{n\pi}{L} x)$$

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Now back to the first equation: $T_n'(t) + c^2 p^2 T_n(t)$, $\Rightarrow$

$$T_n(t) = C_n e^{-c^2 p_n^2 t},$$

We get a family of solutions,
$u_n(x, t) = X_n(x) T_n(t) = B_n \sin(p_n x) e^{-c^2 p_n^2 t}$.

Initial condition: Fourier series

Bright idea (Fourier): combine the elementary solutions in order to meet the boundary condition $u(x,0) = f(x)$, $0 < x < L$.
We look for the solution in the form $u(x,t) = \sum_n u_n(x,t)$ then

$$u(x,0) = \sum_n u_n(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

One can find B_n's from the sine half-range Fourier series for F!
Finally

$$u(x,t) = \sum_n u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-c^2 p_n^2 t}$$

Instruction for cooking the heat equation

- Separate variables and obtain a couple of ordinary differential equation
- Use boundary conditions to find possible values of the parameter
- Find all elementary solutions
- Use Fourier series in order to meet the initial condition
- Check your solution