11.9 Complex Fourier Transfrom (end). Discrete and Fast Fourier Transform (DFT and FFT) 11.7 Fourier Integrals

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Remind: Fourier Transform and Inverse Fourier Transform

We define

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixw} dx$$

$\hat{f}$ is called the Fourier transform of f. Then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{ixw} dw.$$

This is called the inversion formula. It allows to reconstruct the function from its Fourier transform and gives a representation of a function as a combination of the exponential ones.

VERY important example:

$$f(t) = e^{-at^2} \Rightarrow \hat{f}(w) = \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}$$

Idea of proof: $\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-at^2} e{-}iwt dt \Rightarrow$

$$(\hat{f}(w))' = \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} te^{-at^2} e^{-iwt} dt = \frac{i}{2a\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(e^{-at^2}\right)' e^{-iwt} dt =$$

$$-\frac{i}{2a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-at^2} \left(e^{-iwt}\right)' dt = -\frac{w}{2a} \hat{f}(w).$$

Differential equation on $\hat{f}(w) \Rightarrow \hat{f}(w) = Ce^{\frac{-w^2}{4a}}$.
The constant $C = \frac{1}{\sqrt{2a}}$ should be calculated separately(see notes to this lecture)

Convolution

$f(t)$, $g(t)$, $-\infty < t < \infty$; $f, g \to 0$ as $t \to \pm\infty$

Convolution of f and g:

$$h(t) = f * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau$$

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Q1: What is relation between this convolution and one defined when we studied the Laplace transform?

Properties of convolution

- $(f_1 + f_2) * g = f_1 * g + f_2 * g$
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Example (low pass filter): $\phi(t) = \frac{1}{\pi}\frac{\sin \Omega t}{t}$.
We know $\hat{\phi}(w) = 1/\sqrt{2\pi}$ if $|w| < \Omega$ and $\hat{\phi}(w) = 0$ otherwise

$$\mathcal{F}(f * \phi)(w) = \begin{cases} \hat{f}(w), & |w| < \Omega, \\ 0, & \text{otherwise} \end{cases}$$

Discuss notion of *spectrum* of a signal

Discrete Fourier Transform (DFT)

For a signal $f(t)$ $0 < t < 2L$ we have

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}, \quad c_n = \frac{1}{2L} \int_{-L}^{L} f(t) e^{-\frac{in\pi x}{L}} dx$$

But we live in a discrete world !

We know the samples at N points $\{2Lk/N\}$, $k = 0, 1, \ \ldots \ , N-1$ only:

$$f(0), \ f(2L/N), \ , ..., \ f(2L(N-1)/N)$$

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$$\tilde{c}_j = \frac{1}{LN} \sum_{k=0}^{N-1} f\left(\frac{2Lk}{N}\right) e^{-2ikj\pi/N}, \quad j = 0, \pm 1, \pm 2, \ldots$$

This seems to be the replacement for Fourier transform!

Discrete Fourier Transform (DFT) (cont)

We can recover no more than N coefficients from N samples !
Indeed $\tilde{c}_{j+N} = \tilde{c}_j$ We take

$$\tilde{c}_0, \tilde{c}_1, ..., \tilde{c}_{N-1}.$$

Matrix notation:
Input: $f_N = [f(0),\ f(2L/N),\ , ...,\ f(2L(N-1)/N)]^T$
Output $\hat{f}_N = N[\tilde{c}_0 ... \tilde{c}_{N-1}] = \mathcal{F}_N(f_N)$.
To obtain $\tilde{c}_j$ we multiply entries of f_N of by the sequence

$$1,\ w^j,\ w^{2j}, ...,\ w^{(n-1)j}, \quad j = 0, 1, ..., N-1$$

where $w = e^{-2i\pi/N}$.

Final form of DFT

$$\hat{f}_N = F_N f_n$$

where

$$F_N = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & \dots & w^{N-1} \\ \dots & \dots & \dots & \ddots & \dots \\ 1 & w^{N-1} & w^{2(N-1)} & \dots & w^{(N-1)^2} \end{bmatrix}$$

$w = e^{-2i\pi/N}$

F_N - the matrix of Discrete Fourier Transform

This computation requires about N^2 multiplications!

Fast Fourier Transform (FFT)

The Fast Fourier Transform is a family of algorithm which allow to reduce the complexity of computation substantially.
The original idea is to compute the DFT for $N = 2M$ using the symmetry of the matrix F_N. Divide f_N into the even and odd parts $f_e = f_M$ and $f_o = (f(x + 1/N))_M$ and let $\hat{f}_e$ and $\hat{f}_o$ be their Discrete Fourier Transforms (of length M). Then

$$\tilde{c}_j = \frac{1}{2}((\hat{f}_e)_j + w^j \hat{f}_o(j)), \quad j = 0, ..., M-1$$

$$\tilde{c}_{j+M} = \frac{1}{2}((\hat{f}_e)_j - w^j \hat{f}_o(j)), \quad j = 0, ..., M-1$$

FFT requires roughly $N \log_2 N$ operations!

Fourier cosine and sine integrals (Sec. 11.7)

These are analogs of trigonometric Fourier series. Given $f(t)$, $-\infty < t < \infty$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(wt)dt, \;\; B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(wt)dt$$

$$f(t) = \int_0^{\infty} [A(w)\cos(wt) + B(w)\sin(wt)]dw.$$

Half-range expansions

Given $f(t)$, $0 < t < \infty$ we can write

$$f(t) = \int_0^\infty A(w)\cos(wt)dw = f(t) = \int_0^\infty B(w)\sin(wt)]dw,$$

where

$$A(w) = \frac{2}{\pi}\int_0^\infty f(t)\cos(wt)dt, \;\; B(w) = \frac{2}{\pi}\int_0^\infty f(t)\sin(wt)dt$$

Example: Laplace integral

$$f(t) = e^{-kt}, \; t > 0$$