11.4 Approximation by trigonometric polynomials 11.3. Fourier series and forced oscillation

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Trigonometric and exponential Fourier series

Let f be a $2L$-periodic function,then its Fourier series is defined by

$$S_f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right),$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

In complex form it is given by

$$S_f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}, \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(t)e^{-\frac{in\pi x}{L}}dx$$

Complex and real series: example

Let $f(x) = x^2$ on $[0, 1]$ and it is extended to a 1-periodic function.
Compute the Fourier series of f.
We do it in complex form, $c_0 = 1/3$ and for $n \neq 0$

$$c_n = \int_0^1 x^2 e^{-in2\pi x} dx = \frac{i}{2\pi n} + \frac{1}{2\pi^2 n^2}$$

$$S_f(x) = \frac{1}{3} + \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi^2 n^2} + \frac{i}{2\pi n} \right) e^{2\pi inx} =$$

$$= \frac{1}{3} + \sum_{n=1}^{\infty} \frac{\cos 2\pi nx}{\pi^2 n^2} - \sum_{n=1}^{\infty} \frac{\sin 2\pi nx}{\pi n}$$

Norm, Parseval's identity, approximation

The norm of the function from our example is
$\|f\|^2 = \int_0^1 (x^2)^2 dx = \frac{1}{5}$.
By the Parseval's identity

$$\frac{1}{5} = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{9} + 2\sum_{n=1}^{\infty} \left(\frac{1}{4\pi^2 n^2} + \frac{1}{4\pi^4 n^4} \right)$$

If we want to approximate our function $f(x) = x^2$ by a trigonometric polynomial $P_N(x) = \sum_{n=-N}^{N} c_n e^{2\pi i n}$ with the square error

$$E_N = \int_0^1 |f(x) - P_N(x)|^2 dx$$

less then 0.25% of the norm $\|f\|^2$, how large should we choose N?

Application to forced oscillation

Let $f(x) = |x|,\ -\pi \le x \le \pi$ and f is extended to a 2π-periodic function. The Fourier series of f is

$$S_f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2} = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

We want to solve the equation $y'' + \omega^2 y = f(x)$. Let solve it for each term of the form $a_n \cos nx$ by the method of undetermined coefficients. The particular solution is

$$y_n(x) = \frac{a_n}{\omega^2 - n^2}\cos nx$$

Then the solution is given by the series

$$y(x) = \frac{\pi}{2\omega^2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2(\omega^2 - (2k-1)^2)}$$