11.1-11.2. Fourier series

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Remind the basic formulas

Let

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

Then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos ntdt$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin ntdt$$

Today we consider examples on how does this work.

Rectangular wave

Example

Let $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$.

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Then

$$S_f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right) =$$

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$$

Q2: Convergence

Precise statement:

Theorem

Let f be a piece-wise continuous function on $[-\pi, \pi]$ (it can be extended 2π-periodically). Suppose that f has left and right derivatives at each point. Then the Fourier series S_f converges at each point and

$$S_f(x) = f(x-0) + f(x+0),$$

where $f(x \pm 0) = \lim_{h \to 0+} f(x \pm h)$.

Informal statement: If f is "not very bad" continuous 2π periodic function, than its Fourier series convergent.

Attention: Convergence at the end points of the interval

Further examples

Example

Let $f(x) = |x|, \ -\pi \leq x \leq \pi$ and f is extended to a 2π-periodic function. Find the Fourier series of f.

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$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx \ dx = \frac{2}{\pi} \int_{0}^{\pi} x \cos nx \ dx = \ldots = \left. \frac{2 \cos nx}{n^2 \pi} \right|_{x=0}^{x=\pi}$$

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$$a_n = \begin{cases} -\frac{4}{n^2\pi}, & n \ odd \\ 0, & n \ even \end{cases} \quad , \quad b_n = 0$$

$$S_f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$

Odd and even functions

f is odd if $f(x) = -f(-x)$.
Fact: f is odd $\Leftrightarrow$ all $a_n = 0$ (cos coefficients)

f is even if $f(x) = f(-x)$.
Fact: f is even $\Leftrightarrow$ all $b_n = 0$ (sin coefficients)

Odd and even prolongation from $(0.\pi)$
Given a function f on $(0, \pi)$ we can prolongate it either as odd or as even 2π periodic function expand either as sin or cos Fourier series. Which of them is true ?

Half range expansions

Given $f(x)$, $0 < x < \pi$ we have

$$f(x) = a_0 + \sum_1^{\infty} a_n \cos nx = \sum_1^{\infty} b_n \sin nx,$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\ dx,$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\ dx,\ \ n = 1, 2, \dots$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\ dx,\ \ n = 1, 2, \dots .$$

Further examples

Example

$$f(x) = \begin{cases} x + \pi, & -\pi < x < 0, \\ x - \pi, & 0 < x < \pi. \end{cases}$$

Attention: you may evaluate Fourier coefficients along any interval whose length is the period. Sometimes It may be convenient to chose appropriate interval.
Sometimes you need not evaluate integrals order to find the Fourier coefficients:
Find the Fourier series for $f(x) = (\cos x + \sin x)^2$

Period $2L$

Given $f(x)$ such that $f(x) = f(x + 2L)$. The corresponding Fourier series has the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\ dx,$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x\ dx, \ \ n = 1, 2, \ldots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x\ dx, \ \ n = 1, 2, \ldots.$$

Exercises:

- Write the formulae for half range expansions for functions on $(0, L)$
- Write the formulae for complex Fourier series for $2L$ periodic functions