11.3-11.4. Complex Fourier series, Least square method, Forced oscillations

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Complex Fourier series

Remind what we know: f is a 2π-periodic function:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \quad (*)$$

and also

$$\sin nt = \frac{e^{int} - e^{-int}}{2i}, \ \cos nt = \frac{e^{int} + e^{-int}}{2}$$

If we substitute these expressions in (*) we obtain
Complex Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

Formulas for coefficients

Maybe obtained through a_n's, b_n's of course.
But:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt.$$

Idea is the same: *orthogonality*.The orthogonality relation:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}e^{-imt}\ dt = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}$$

Comment Just one formula for the coefficients and just one orthogonality formula.

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Convergence rules: the same as for trigonometic series, because this is the same series, just another representation.

Example (from the textbook)

Let $f(t) = e^{at}$, $-\pi < t < \pi$.

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{at}e^{-int}dt = ... = \frac{(-1)^n}{\pi(a - in)}\sinh a\pi.$$

Respectively

$$e^{at} = \frac{\sinh a\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a - in}e^{int}.$$

We can obtain usual trigonometric Fourier series from this expression: .....

Once again about orthogonality

Review of what we know in finite dimensions:

A. Real variable version:

- Space $\mathbf{R}^n = \{\vec{x} = (x_j)_{j=1}^n, x_j \in \mathbf{R}\}$
- Inner product and length
  $\langle \vec{x}, \vec{y} \rangle = \sum_1^n x_j y_j$, $\|x\|^2 = \langle \vec{x}, \vec{x} \rangle = \sum_1^n |x_j|^2$.
- Orthonormal basis and expansion:
  $\{\vec{e}_j\}_{j=1}^n \subset \mathbf{R}^n$, $\vec{e}_j = (\underbrace{0, \ldots, 0, 1}_{j}, 0, \ldots, 0\}$.
  $\vec{x} = \sum_{j=1}^n c_j \vec{e}_j$, $c_j = x_j = \langle \vec{x}, \vec{e}_j \rangle$
- Projection of $\vec{x}$ on the plane Π_k, generated by $\{\vec{e}_j\}_{j=1}^k$, $k < n$:
  $$P_k \vec{x} = \sum_{j=1}^k c_j \vec{e}_j$$
- Distance between $\vec{x}$ and Π_k:
  $\mathrm{dist}(\vec{x}, \Pi_k)^2 = \|\vec{x} - P_k \vec{x}\|^2 = \sum_{j=k+1}^n |x_j|^2$

Once again about orthogonality

Review of what we know in finite dimensions:

A. Complex variable version:

- Space $\mathbf{C}^n = \{\vec{x} = (x_j)_{j=1}^n, x_j \in \mathbf{C}\}$
- Inner product and length
  $\langle \vec{x}, \vec{y} \rangle = \sum_1^n x_j \overline{y_j}$, $\|x\|^2 = \langle \vec{x}, \vec{x} \rangle = \sum_1^n |x_j|^2$.
- Orthonormal basis and expansion:
  $\{\vec{e}_j\}_{j=1}^n \subset \mathbf{C}^n$, $\vec{e}_j = \{\underbrace{0, \ldots, 0, 1}_{j}, 0, \ldots, 0\}$.
  $\vec{x} = \sum_{j=1}^n c_j \vec{e}_j$, $c_j = x_j = \langle \vec{x}, \vec{e}_j \rangle$, now c_j are complex numbers.
- Projection of $\vec{x}$ on the plane Π_k, generated by $\{\vec{e}_j\}_{j=1}^k$, $k < n$:
  $$P_k \vec{x} = \sum_{j=1}^k c_j \vec{e}_j$$
- Distance between $\vec{x}$ and Π_k:
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Orthogonality and Fourier series

Functions on $(-\pi, \pi)$ from a vector space, we can add them, multiply by constants etc. This space is now infinite dimensional, t plays the role of n to some extend.

we want to modify the same construction for this space !

Q1: Given $f(t)$, how can we measure the "length" of the function $f(t)$, $-\pi < t < \pi$?

Norm (="length") of f is

$$\|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2 dt.$$

Motivation:

- similarity with finite dimensional case
- energy motivation
- noise reduction
- least square method

Q2: How we chose inner product?
We have have $\|f\|^2 = \langle f, f \rangle$. That is why we define

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt.$$

Q3: How we chose orthogonal basis?

$$\vec{e}_n = e^{int}, \;\; n = 0, \pm 1, \pm 2, \;\ldots$$

Orthogonality relations take the form

$$\langle \vec{e}_k, \vec{e}_l \rangle = \begin{cases} 0, & k \neq l, \\ 1, & k = l. \end{cases}$$

Orthogonality and Fourier series, ctd

Q4: How do expansions look like?

$$f = \sum_{n=-\infty}^{\infty} c_n \vec{e}_n, \;\; c_n = \langle f, \vec{e}_n \rangle.$$

Put $\vec{e}_n = e^{int}$, then $c_n = c_n(f) \langle f, \vec{e}_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e{-}intdt$ and we arrive to complex Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n(f) e^{int}$$

Q5: How do we measure length in terms of the expansion coefficients ?
Parseval identity: (just Pythagorus theorem for infinite dimensional case)

$$\|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Exponential polynomials of degree k is the set Π_k of all finite sums of the form

$$s(t) = \sum_{n=-k}^{k} c_n e^{int}$$

Approximation of f by exponential polynomials (distance from f to Π_k):

For each $s \in \Pi_k$ we have

$$\|s - f\|^2 \geq \sum_{|n|>k} |c_n(f)|^2,$$

equality takes place for truncated Fourier series

$$s(t) = s_f(t) = \sum_{|n|\leq k} c_n(f) e^{int},$$

Example (2011 exam problem)

$$f(t) = \begin{cases} 0, & -\pi < t < 0 \\ 1, & 0 < t < \pi \end{cases}$$

Find the Fourier series and then find the sum $\sum_{n=1}^{\infty} 1/(2n-1)^2$

Answers:

$$f(t) = \frac{1}{2} + \frac{1}{i\pi} \sum_{l=-\infty}^{\infty} \frac{1}{2l-1} e^{i(2l-1)t}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Digression:

- various way to calculate answer
- various types of convergence