11.1-11.2. Fourier series

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Periodic functions

Function f defined on the *whole* real axis has period p if

$$f(t) = f(t + p) \text{ for all } t \in \mathbf{R}$$

Properties

- If f and g have period p then so does their linear combinations $af(x + p) + bg(x + p) = af(x) + bg(x)$.
- $f(t + np) = f(t)$ for all integer $n \Rightarrow$
  Basic period i the smallest possible period for a given function
- $f(t)$ has period $p \Rightarrow g(t) = f(ct)$ has period p/c.
  Therefore it suffices to study the functions of some fixed period. Then we can make scaling. We will deal mainly with functions of period 2π.
- A periodic function with period p can be reconstructed from its values on a segment of length p

Examples

<u>In nature</u>

- Pendulum
- Wave motion
- Cristal structures
- Sound waves.

<u>In mathematics</u>

- $1, \ \sin nt, \ \cos nt \ \ n = 1, 2, 3, \dots$ - trigonometric system
- $e^{int}, \ n = 0, 1, 3, \dots$ - exponential system

They are related by the Euler formulas

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$

or

$$\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}, \ \cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$$

Refresh your knowledge on complex numbers !!

Basic fact

Each 2π-periodic function f can be represented as a series in trigonometrical or in exponential system

Trigonometrical Fourier series:

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

Exponential (complex) Fourier series:

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{int}.$$

Basic question

Q1: How does one find the coefficients a_n, b_n or c_n?

Q2: When and in what sense does the Fourier series converge to the function f?

We answer these question starting from the trigonometrical Fourier series

Q1: Coefficients of the trigonometrical Fourier series

Basic formulas:

Let

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

Then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt dt$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt dt$$

Idea of the proof: ORTHOGONALITY.

Orthogonality

Model: $\vec{x} \in \mathbf{R}^3$, $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$. Then $x_j = \langle \vec{x}, \vec{e}_j \rangle$.
This is because

$$\langle \vec{e}_j, \vec{e}_l \rangle = \begin{cases} 0, & l \neq j \\ 1, & l = j. \end{cases}$$

Ortogonality relations for Fourier series:

$$\int_{-\pi}^{\pi} \cos nx \cos mx \ dx = 0, m \neq m;$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx \ dx = 0;$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \ dx = 0, m \neq m;$$

$$\int_{-\pi}^{\pi} \cos^2 nx \ dx = 0 = \int_{-\pi}^{\pi} \sin^2 nx dx = \pi.$$

Now it is easy to obtain expressions for the coefficients ...

Fourier series

Given a 2π-periodic function f, consider *its Fourier series*

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nxdx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nxdx$$

Rectangular wave

Example

Let $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$.

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Let $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$.

Then

$$S_f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right) =$$

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$$

Q2: Convergence

Precise statement:

Theorem

Let f be a piece-wise continuous function on $[-\pi, \pi]$ (it can be extended 2π-periodically). Suppose that f has left and right derivatives at each point. Then the Fourier series S_f converges at each point and

$$S_f(x) = f(x-0) + f(x+0),$$

where $f(x \pm 0) = \lim_{h \to 0+} f(x \pm h)$.

Informal statement: If f is "not very bad" continuous 2π periodic function, than its Fourier series convergent.

Attention: we need continuity on the *whole* real axis.

Further examples

Example

Let $f(x) = |x|, \ -\pi \leq x \leq \pi$ and f is extended to a 2π-periodic function. Find the Fourier series of f.

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$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx \; dx = \frac{2}{\pi}\int_0^{\pi} x \cos nx \; dx = \ldots = \left.\frac{2\cos nx}{n^2\pi}\right|_{x=0}^{x=\pi}$$

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$$a_n = \begin{cases} -\frac{4}{n^2\pi}, & n\ odd \\ 0, & n\ even \end{cases} \quad , \quad b_n = 0$$

$$S_f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

Further examples

Example

$$f(x) = \begin{cases} t + \pi, & -\pi < t < 0, \\ t - \pi, & 0 < t < \pi. \end{cases}$$

Attention: you may evaluate Fourier coefficients along any interval whose length is the period. Sometimes It may be convenient to chose appropriate interval.

Dirty exam trick: Find the Fourier series for $f(x) = (\cos x + \sin x)^2$

Odd and even functions

f is odd if $f(x) = -f(-x)$.
Fact: f is odd $\Leftrightarrow$ all $a_n = 0$ (cos coefficients)

f is even if $f(x) = f(-x)$.
Fact: f is even $\Leftrightarrow$ all $b_n = 0$ (sin coefficients)

Given a function f on $(0, \pi)$ we can prolongate it either as odd or as even 2π periodic function expand either as sin or cos Fourier series. Which of them is true ?
Half range expansions.

Further exercises

Exercises:

1. Produce formulas for Fourier coefficients for half range expansions

2. Write basic formulas for the Fourier series for functions with period with period $2L$.