6.5 Applications of the convolution theorem
6.6 Differentiation/integration of the Laplace transform
6.7 Systems of ODE

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Example
Solve the initial value problem

\[ y'' + 9y = g(t), \quad y(0) = -1, \; y'(0) = 9 \]

We apply the Laplace transform:

\[ Y(s) = \frac{G(s)}{s^2 + 9} - \frac{s}{s^2 + 9} + \frac{9}{s^2 + 9} \]

where \( G \) is the Laplace transform of \( g \) and then use the convolution theorem

\[ y(t) = \frac{1}{3} \int_{0}^{t} g(\tau) \sin 3(t - \tau) \, d\tau - \cos 3t + 3 \sin 3t \]

Compare the answer to the one given by the method of variation of parameters.
Example

( an exam problem) Find $y(t)$ that solves the equation

$$y(t) + \int_0^t y(t - \tau)e^\tau d\tau = \delta(t - 5), \quad t > 0.$$ 

We apply the Laplace transform

$$Y(s)(1 + \frac{1}{s - 1}) = e^{-5s}, \quad Y(s) = (1 - 1/s)e^{-5s}$$

Then, taking the inverse Laplace transform we get:

$$y(t) = \delta(t - 5) - u(t - 5)$$
Differentiation and integration of Laplace transforms

Differentiation: \( F'(s) = -\mathcal{L}(tf(t)) \)

Integration: \( \int_s^\infty F(\sigma)d\sigma = \mathcal{L}\left(\frac{1}{t}f(t)\right) \)

Example

Find \( \mathcal{L}^{-1}\left(\log \left(1 + \frac{\omega^2}{s^2}\right)\right) \)

\[
F(s) = \log \left(1 + \frac{\omega^2}{s^2}\right) = \mathcal{L}(f(t)) \quad \Rightarrow
\]

\[
F'(s) = \frac{2s}{s^2 + \omega^2} - \frac{2}{s} = -\mathcal{L}(tf(t)) \quad \Rightarrow
\]

\[
-tf(t) = 2(\cos \omega t - 1) \quad \Rightarrow \quad f(t) = \frac{2}{t}(1 - \cos \omega t)
\]
Application to some ODE with non-constant coefficients

The differentiation rule implies

\[ \mathcal{L}(ty') = -(\mathcal{L}(y'))' = (sY - y(0))' = sY' + Y \]
\[ \mathcal{L}(ty'') = -(\mathcal{L}(y''))' = -(s^2 Y - sy(0) - y'(0))' = -s^2 Y' - 2sY + y(0) \]

Some second order equations in \( y \) are reduced to first order in \( Y \).

**Example (Laguerre’s Equation)**

\( ty'' + (1 - t)y' + ny = 0 \) is reduced to

\[ (s - s^2)Y' + (n + 1 - s)Y = 0 \]

It is first order separable ODE, \( Y_n = (s - 1)^n s^{n-1} \). The inverse Laplace transform gives

\[ y_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n} (t^n e^{-t}) \]
Systems of ODEs

\[
y'_1(t) = a_{11}y_1(t) + a_{12}y_2(t) + g_1(t)
y'_2(t) = a_{21}y_1(t) + a_{22}y_2(t) + g_2(t)
y_1(0) = y_1, \ y_2(0) = y_2.
\]

- Apply the Laplace transform to get a system of equations on \( Y_1(s), \ Y_2(s) \)
- solve this system and find \( Y_1(s), \ Y_2(s) \);
- take the inverse Laplace transform
Example

An electrical network (6.7.19), \( i_1(0) = i_2(0) = 0, \)

\[ 4i_1 + 8(i_1 - i_2) + 2i'_1 = 390 \cos t, \quad 8i_2 + 4i'_2 + 8(i_2 - i_1) = 0 \]

Applying the Laplace transform, we get

\[ 12l_1 - 8l_2 + 2sl_1 = 390s(s^2 + 1)^{-1}, \quad 16l_2 - 8l_1 + 4sl_2 = 0 \]

Then \( 2l_1 = (s + 4)l_2 \) and

\[ [(s + 6)(s + 4) - 8]l_2 = 390s(s^2 + 1)^{-1} \]

\[ l_2 = \frac{390s}{(s^2 + 1)(s + 2)(s + 8)}, \quad l_1 = \frac{195s(s + 4)}{(s^2 + 1)(s + 2)(s + 8)} \]

...partial fractions + inverse Laplace
Computations for the last example

From MATLAB residue:
\[
\frac{390s}{(s^2+1)(s+2)(s+8)} = \frac{8}{s+8} - \frac{26}{s+2} + \frac{9-6i}{s-i} + \frac{9+6i}{s+i}
\]
\[
\frac{390s}{(s^2+1)(s+2)(s+8)} = \frac{8}{s+8} - \frac{26}{s+2} + \frac{18s+12}{s^2+1}
\]

Then the inverse Laplace transform gives:
\[
i_2(t) = 8e^{-8t} - 26e^{-2t} + 18 \cos t + 12 \sin t
\]

Similarly:
\[
\frac{195s^2+780s}{(s^2+1)(s+2)(s+8)} = -\frac{16}{s+8} - \frac{26}{s+2} + \frac{21-7.5i}{s-i} + \frac{21+7.5i}{s+i}
\]
\[
\frac{195s^2+780s}{(s^2+1)(s+2)(s+8)} = -\frac{16}{s+8} - \frac{26}{s+2} + \frac{42s+15}{s^2+1}
\]
\[
i_1(t) = -26e^{-8t} - 26e^{-2t} + 42 \cos t + 15 \sin t
\]