6.5 - 6.7. Convolutioin (end) Differentiation/integration of L.t. Systems of ODE

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Reminder

$$\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-ts}dt$$

$$f * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

$$\mathcal{L}(f * g)(s) = \mathcal{L}(f)(s)\mathcal{L}(g)(s).$$

Application #1

You know

$$\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}F(s)$$

This can be now viewed as convolution rule since

$$\int_0^t f(\tau)d\tau) = \frac{1}{s}F(s) = f * u(t)$$

Remark: we know that if $f(0) = 0$ then $\mathcal{L}(f)(s) = sF(s)$. Apply this knowledge to what we know:

$$\mathcal{L}u(t-c) = \frac{1}{s}e^{-tc}, \quad \mathcal{L}\delta(t-c) = e^{-tc}$$

It looks as $\delta(t-c) = (u(t-c))'$. Can you give another interpretation of this equality ?

More examples

Example

Find the Laplace transform of $f(t) = \int_0^t (t-\tau)^3 \sin(2\tau) d\tau$.

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More examples

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Example

Find the inverse Laplace transform of $F(s) = \frac{1}{s^3-s^2}$

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Example

Find the inverse Laplace transform of $F(s) = \frac{1}{s^3-s^2}$

$$\mathcal{L}^{-1}\{(s^3-s^2)^{-1}\} = \mathcal{L}^{-1}\{s^{-2}\} * \mathcal{L}^{-1}\{(s-1)^{-1}\} = t * e^t = \int_0^t e^\tau (t-\tau) d\tau$$

$$= t(e^t - 1) - (te^t - e^t + 1) = e^t - t - 1$$

Another viewpoint on ODE

Initial value problem:

$$ay'' + by' + cy = g, \quad y(0) = K_0,\ y'(0) = K_1$$

$$Y(s) = \underbrace{\frac{G(s)}{as^2 + bs + c}}_{\text{contribution of the right-hand side}} + \underbrace{\frac{(as+b)K_0 + aK_1}{as^2 + bs + c}}_{\text{contribution of the initial conditions}}$$

Transfer function

$$Q(s) = \frac{1}{as^2 + bs + c},\ q(t) = \mathcal{L}^{-1}Q(t)$$

Then

$$y(t) = \mathcal{L}\left(\frac{(as+b)K_0 + aK_1}{as^2 + bs + c}\right) + g * q(t).$$

Example of an exam problems

Example

Solve the initial value problem

$$y'' + 9y = g(t), \quad y(0) = -1, y'(0) = 9$$

We find the Laplace transform of y

$$Y(s) = \frac{G(s)}{s^2+9} - \frac{s}{s^2+9} + \frac{9}{s^2+9}$$

where G is the Laplace transform og g and then use the convolution theorem

$$y(t) = \frac{1}{3}\int_0^\tau g(\tau)\sin 3(t-\tau)d\tau - \cos 3t + 3\sin 3t$$

Compare the answer to the one given by the method of variation of parameters.

Integral equations

Example from old exam:

$$y(t) + \int_0^t y(t-\tau)e^{\tau}\,d\tau = \delta(t-5),\ \ t > 0.$$

$$Y(s) = \frac{s-1}{s}e^{-5s};$$

$$y(t) = \delta(t-5) - u(t-5)$$

Remark: You have to be able to recognise convolution.

Differentiation and integration of L.t.

<u>Differentiation:</u> $F'(s) = -\mathcal{L}(tf(t))$ - obvious.

<u>Integration:</u> $\int_s^\infty F(\sigma)d\sigma = \mathcal{L}(\frac{1}{t}f(t))$

Discussion: Laplace transform decays as $s \to \infty$.

Example: Find $\mathcal{L}^{-1}\left(\log\left(1+\frac{\omega^2}{s^2}\right)\right)$

$$F(s) = \log\left(1+\frac{\omega^2}{s^2}\right) = \mathcal{L}(f(t)) \;\Rightarrow$$

$$F'(s) = \frac{2s}{s^2+\omega^2} - \frac{2}{s} = -\mathcal{L}(tf(t)) \;\Rightarrow$$

$$-tf(t) = 2(\cos\omega t - 2) \;\Rightarrow f(t) = \frac{2}{t}(1-\cos\omega t)$$

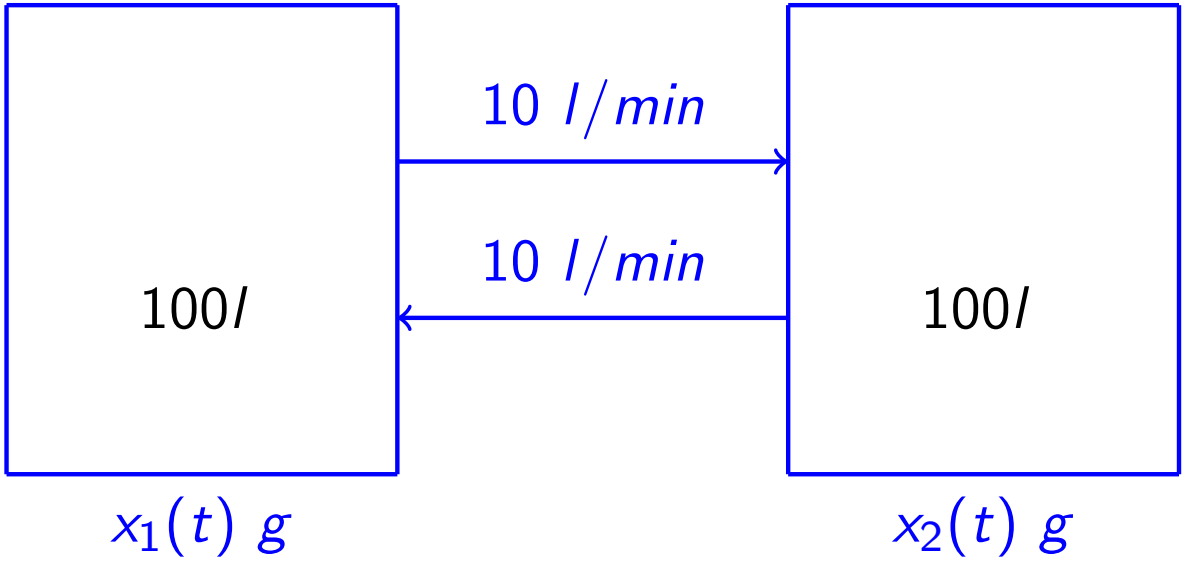
Systems of ODE

$$y_1'(t) = a_{11}y_1(t) + a_{12}y_2(t) + g_1(t)$$
$$y_2'(t) = a_{21}y_1(t) + a_{22}y_2(t) + g_2(t)$$
$$y_1(0) = y_1,\ y_2(0) = y_2.$$

Idea: *business as usual:*
- Laplace transform everything get a system go <u>linear</u> equations on $Y - 1(s),\ Y_2(s)$;
- solve this system and find $Y_1(s),\ Y_2(s)$; - make inverse Laplace transform

Boring example

Two connected tanks:



Suppose that initially the first tank contains 30 g of salt and the second contains pure water. The solutions are mixing at a rate of 10 l/min (it is the rate of flow in each direction). Find $x_1(t)$ and $x_2(t)$, amounts of salt (in g) in the first and second tanks at moment t.

Example (ctd):

$$x_1' = 0.1x_2 - 0.1x_1$$
$$x_2' = 0.1x_1 - 0.1x_2$$
$$x_1(0) = 30, \ x_2(0) = 0$$

The corresponding system for the Laplace transforms:

$$(s + 0.1)X_1(s) - 0.1X_2(s) = 30$$
$$-0.1X_1(s) + (s + 0.1)X_2(s) = 0$$

Discuss:

- reduction to the second order equation
- matrix form