6.5 - 6.7. Convolution (end)
Differentiation/integration of L.t.
Systems of ODE

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\[ \mathcal{L}(f)(s) = \int_{0}^{\infty} f(t) e^{-ts} \, dt \]

\[ f \ast g(t) = \int_{0}^{t} f(t - \tau) g(\tau) \, d\tau \]

\[ \mathcal{L}(f \ast g)(s) = \mathcal{L}(f)(s) \mathcal{L}(g)(s). \]
Application #1

You know

\[ \mathcal{L} \left( \int_0^t f(\tau) d\tau \right) = \frac{1}{s} F(s) \]

This can be now viewed as convolution rule since

\[ \int_0^t f(\tau) d\tau = \frac{1}{s} F(s) = f \ast u(t) \]

Remark: we know that if \( f(0) = 0 \) then \( \mathcal{L}(f)(s) = sF(s) \). Apply this knowledge to what we know:

\[ \mathcal{L}u(t - c) = \frac{1}{s} e^{-tc}, \quad \mathcal{L}\delta(t - c) = e^{-tc} \]

It looks as \( \delta(t - c) = (u(t - c))' \). Can you give another interpretation of this equality?
Example
Find the Laplace transform of \( f(t) = \int_0^t (t - \tau)^3 \sin(2\tau) d\tau \).
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More examples

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Example

Find the inverse Laplace transform of \( F(s) = \frac{1}{s^3 - s^2} \).
More examples

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\[
\mathcal{L}^{-1}\{ (s^3-s^2)^{-1} \} = \mathcal{L}^{-1}\{ s^{-2} \} \ast \mathcal{L}^{-1}\{ (s-1)^{-1} \} = t \ast e^t = \int_0^t e^\tau (t-\tau) d\tau
\]

\[
= t(e^t - 1) - (te^t - e^t + 1) = e^t - t - 1
\]
Another viewpoint on ODE

Initial value problem:

\[ ay'' + by' + cy = g, \quad y(0) = K_0, \quad y'(0) = K_1 \]

\[ Y(s) = \frac{G(s)}{as^2 + bs + c} + \frac{(as + b)K_0 + aK_1}{as^2 + bs + c} \]

Transfer function

\[ Q(s) = \frac{1}{as^2 + bs + c}, \quad q(t) = \mathcal{L}^{-1}Q(t) \]

Then

\[ y(t) = \mathcal{L} \left( \frac{(as + b)K_0 + aK_1}{as^2 + bs + c} \right) + g \ast q(t). \]
Example of an exam problems

Example

Solve the initial value problem

\[ y'' + 9y = g(t), \quad y(0) = -1, \quad y'(0) = 9 \]

We find the Laplace transform of \( y \)

\[ Y(s) = \frac{G(s)}{s^2 + 9} - \frac{s}{s^2 + 9} + \frac{9}{s^2 + 9} \]

where \( G \) is the Laplace transform of \( g \) and then use the convolution theorem

\[ y(t) = \frac{1}{3} \int_0^t g(\tau) \sin 3(t - \tau) \, d\tau - \cos 3t + 3 \sin 3t \]

Compare the answer to the one given by the method of variation of parameters.
Example from old exam:

\[ y(t) + \int_0^t y(t - \tau) e^{\tau} d\tau = \delta(t - 5), \quad t > 0. \]

\[ Y(s) = \frac{s - 1}{s} e^{-5s}; \]

\[ y(t) = \delta(t - 5) - u(t - 5) \]

Remark: You have to be able to recognise convolution.
Differentiation and integration of L.t.

**Differentiation:** \( F'(s) = -\mathcal{L}(tf(t)) \) - obvious.

**Integration:** \( \int_s^\infty F(\sigma)d\sigma = \mathcal{L}\left(\frac{1}{t}f(t)\right) \)

**Discussion:** Laplace transform decays as \( s \to \infty \).

**Example:** Find \( \mathcal{L}^{-1}\left(\log\left(1 + \frac{\omega^2}{s^2}\right)\right) \)

\[
F(s) = \log\left(1 + \frac{\omega^2}{s^2}\right) = \mathcal{L}(f(t)) \quad \Rightarrow \\
F'(s) = \frac{2s}{s^2 + \omega^2} - \frac{2}{s} = -\mathcal{L}(tf(t)) \quad \Rightarrow \\
-\frac{t}{f(t)} = 2(\cos \omega t - 2) \quad \Rightarrow f(t) = \frac{2}{t}(1 - \cos \omega t) \]
Systems of ODE

\[
y_1'(t) = a_{11}y_1(t) + a_{12}y_2(t) + g_1(t) \\
y_2'(t) = a_{21}y_1(t) + a_{22}y_2(t) + g_2(t) \\
y_1(0) = y_1, \ y_2(0) = y_2.
\]

Idea: *business as usual*:
- Laplace transform everything get a system go linear equations on \( Y - 1(s), \ Y_2(s) \);
- solve this system and find \( Y_1(s), \ Y_2(s) \); - make inverse Laplace transform
Suppose that initially the first tank contains 30 g of salt and the second contains pure water. The solutions are mixing at a rate of 10 l/min (it is the rate of flow in each direction). Find $x_1(t)$ and $x_2(t)$, amounts of salt (in g) in the first and second tanks at moment $t$. 
Example (ctd):

\[ x'_1 = 0.1x_2 - 0.1x_1 \]
\[ x'_2 = 0.1x_1 - 0.1x_2 \]
\[ x_1(0) = 30, \quad x_2(0) = 0 \]

The corresponding system for the Laplace transforms:

\[ (s + 0.1)X_1(s) - 0.1X_2(s) = 30 \]
\[ -0.1X_1(s) + (s + 0.1)X_2(s) = 0 \]

Discuss:

- reduction to the second order equation
- matrix form