6.4 Short impulses. Dirac $\delta$-function.
6.5 Convolution

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Physical meaning of 2nd order differential equation

\[ my''(t) + ky'(t) + \gamma y(t) = f(t), \quad y(0) = y_0, \quad y'(0) = y_1. \]

- \( m \) - mass;
- \( k \) - resistance (friction for example);
- \( \gamma \) - elasticity;
- \( f \) - external force;
- \( y_0, y_1 \) - initial position and velocity.

\text{Impulse: } \int_0^\infty f(t)dt

\text{Impulse - "measure of efforts made by the external force"}
Hammer blow = instant hit with given impulse (for example impulse equal 1)

*Idea:* Take a very short hit of impulse 1, let its length approach zero and see what is going to happen

We model an instant hit at the moment $t_0$. Take

$$f_\Delta(t) = \begin{cases} 
0, & t < t_0; \\
\frac{1}{\Delta}, & t_0 \leq t < t_0 + \Delta; \\
0, & t_0 + \Delta < t.
\end{cases}$$

and then pass to the limit as $\Delta \to 0$. 
Bad news and good news

Unfortunately

\[ \lim_{\Delta \to 0} f_{\Delta}(t) = \begin{cases} \infty, & t = t_0, \\ 0, & t \neq t_0 \end{cases} \]

does not make much sense.

Fortunately

\[ \lim_{\Delta \to 0} (\mathcal{L} f_{\Delta})(s) \]

perfectly exists!

Hence we can use it in the right-hand side when solving ODE by using Laplace transform!
We consider the limit of those functions \( f_\Delta \) as \( \Delta \to 0 \). This limit is called the Dirac delta function,

\[
\delta_{t_0}(t) = \delta(t - t_0)
\]

which is zero unless \( t = t_0 \) and satisfies

\[
\int_{-\infty}^{\infty} \delta_{t_0}(t) \, dt = 1.
\]
We can compute integrals of \( \delta(t - t_0)g(t) \)

\[
\int_{-\infty}^{\infty} \delta(t - t_0)g(t)\,dt = \lim_{\Delta \to 0} \frac{1}{\Delta} \int_{t_0}^{t_0 + \Delta} g(t)\,dt = g(t_0),
\]

where \( g \) is a continuous function.

In particular, if \( t_0 > 0 \) then

\[
\mathcal{L}\{\delta_{t_0}\}(s) = \int_{0}^{\infty} e^{-st} \delta(t - t_0)\,dt = e^{-st_0}
\]

When \( t_0 = 0 \) we write \( \delta_0 = \delta \) and we will agree that \( \mathcal{L}\{\delta\} = 1 \).

We can compute the anti-derivative of \( \delta_{t_0} \)

\[
\int_{-\infty}^{t} \delta(t - t_0)\,dt = \begin{cases} 
1, & t > t_0 \\
0, & t < t_0
\end{cases} = u_{t_0}(t)
\]
Example (old exam problem):

\[ y''(t) + 100y(t) = \delta(t - 2), \ y(0) = 0, \ y'(0) = 0. \]

Test question: can you find \( y(1) \) without solving the equation?

\[ \mathcal{L} : y \mapsto Y(s), \ y'' \mapsto s^2 Y(s), \ \delta(t - 2) \mapsto e^{-2s} \]

\[ Y(s)(s^2 + 100) = e^{-2s}; \ Y(s) = \frac{e^{-2s}}{s^2 + 100}. \]

\[ y(t) = \mathcal{L}^{-1} \left( \frac{e^{-2s}}{s^2 + 100} \right) = 0.1 u(t - 2) \sin 10(t - 2). \]
Let $f$ and $g$ be two piece-wise continuous functions on $[0, +\infty)$ we define a new function

$$h(t) = (f \ast g)(t) = \int_0^t f(t - \tau)g(\tau) \, d\tau = \int_0^t f(\tau)g(t - \tau) \, d\tau$$

It is called the convolution of $f$ and $g$. Basic rules for convolutions are

- $f \ast g = g \ast f$
- $f \ast (ag) = a(f \ast g)$ when $a$ is a constant
- $f \ast (g_1 + g_2) = f \ast g_1 + f \ast g_2$
- $(f \ast g) \ast w = f \ast (g \ast w)$
Theorem

Suppose that $f$ and $g$ are piece-wise continuous functions and there Laplace transforms are defined when $s > a$, $\mathcal{L}\{f\} = F$, $\mathcal{L}\{g\} = G$. Then the Laplace transform of their convolution $f \ast g$ is also defined when $s > a$ and

$$\mathcal{L}\{f \ast g\}(s) = F(s)G(s)$$
Convolution: examples

1. \( f(t) = t, \ g(t) = t^2, \ f \ast g = \int_0^t (t - \tau)\tau^2 d\tau = t^4/12 \)
   Check the convolution theorem
   \( \mathcal{L}\{t\} = s^{-2}, \ \mathcal{L}\{t^2\} = 2s^{-3} \) and \( \mathcal{L}\{t^4/12\} = (24s^{-5}) = 2s^{-5} \).

2. \( f(t) = \cos t, \ g(t) = 1, \ f \ast g = \int_0^t \cos \tau d\tau = \sin t \)
   The Laplace transforms are:
   \( \mathcal{L}\{\cos t\} = \frac{s}{s^2+1}, \ \mathcal{L}\{1\} = \frac{1}{s} \) and \( \mathcal{L}\{\sin t\} = \frac{1}{s^2+1} \).

3. \( g(t) = 1, \ f \ast g = \int_0^t f(\tau)d\tau \) and

\[
\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \frac{1}{s} \mathcal{L}\{f\}(s)
\]
Applications of the convolution theorem, examples

Example
Find the Laplace transform of \( f(t) = \int_0^t (t - \tau)^3 \sin(2\tau) d\tau \).

\[
\mathcal{L}\{f\}(s) = \mathcal{L}\{t^3\}(s)\mathcal{L}\{\sin 2t\}(s) = \frac{6}{s^4} \cdot \frac{2}{s^2 + 4} = \frac{12}{s^6 + 4s^4}
\]

Example
Find the inverse Laplace transform of \( F(s) = \frac{1}{s^3 - s^2} \)

\[
\mathcal{L}^{-1}\{(s^3 - s^2)^{-1}\} = \mathcal{L}^{-1}\{s^{-2}\} \ast \mathcal{L}^{-1}\{(s-1)^{-1}\} = t \ast e^t = \int_0^t e^\tau (t-\tau) d\tau
\]

\[
(= t(e^t - 1) - (te^t - e^t + 1) = e^t - t - 1)
\]