6.4 / 6.5. Short impulses. Dirac δ-function. Convolution (beginning)

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Physical meaning of 2nd order differential equation

\[ my''(t) + ky'(t) + \gamma y(t) = f(t), \ y(0) = y_0, y'(0) = y_1. \]

\( m \) - mass;
\( k \) - resistance (friction for example);
\( \gamma \) - elasticity;
\( f \) - external force;
\( y_0, y_1 \) - initial position and velocity.

**Impulse:** \( \int_0^\infty f(t) \, dt \)

Impulse - "measure of efforts made by the external force"
Modelling instant hit (hummerblow):

Hummer blow = instant hit with given impulse (for example impulse equal 1)

Idea: Take a very short hit of impulse 1, let its length approach zero and see what is going to happen
Let we be modelling an instant hit at the moment $t_0$. Then we may take

$$f_\Delta(t) = \begin{cases} 
0, & t < t_0; \\
\frac{1}{\Delta}, & t_0 \leq t < t_0 + \Delta; \\
0, & t_0 + \Delta < t.
\end{cases}$$

and then pass to the limit as $\Delta \to 0$. 

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and then pass to the limit as $\Delta \to 0$. 
We consider the limit of those functions $f_\Delta$ as $\Delta \to 0$. This limit is called the Dirac delta function,

$$\delta_{t_0}(t) = \delta(t - t_0)$$

which is zero unless $t = t_0$ and satisfies

$$\int_{-\infty}^{\infty} \delta_{t_0}(t)dt = 1.$$
Bad news and good news

Unfortunately

\[ \lim_{\Delta \to 0} f_\Delta(t) = \begin{cases} \infty, & t = t_0, \\ 0, & t \neq t_0 \end{cases} \]

does not make much sense.

Fortunately \[ \lim_{\Delta \to 0} (L f_\Delta)(s) \] perfectly exists!

Hence we can use it in the right-hand side when solving ODE by using Laplace transform!
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Hence we can use it in the right-hand side when solving ODE by using Laplace transform!
Calculating the Laplace transform:

1. \( f_\Delta(t) = \frac{1}{\Delta} (u_{t_0}(t) - u_{t_0+\Delta}(t)) \) so one can find \((\mathcal{L}f_\Delta)(s)\) by using the 2nd shift theorem.

2. But it is so easy to find it by bare hands!

\[
(\mathcal{L}f_\Delta)(s) = \frac{1}{\Delta} \int_{t_0}^{t_0+\Delta} e^{-ts} dt = -\frac{1}{\Delta s} e^{-ts} \bigg|_{t_0}^{t_0+\Delta} =
\]

\[
-\frac{1}{\Delta s} \left[ e^{-(t_0+\Delta)s} - e^{-t_0s} \right] = e^{-t_0s} \frac{e^{-\Delta s} - 1}{-\Delta s} \rightarrow 1 \text{ as } \Delta \rightarrow 0
\]

Finally

\[
(\mathcal{L}f_\Delta)(s) \rightarrow e^{-t_0s} \text{ as } \Delta \rightarrow 0
\]
Definition

Dirac delta-function located at $t_0$

$$\delta_{t_0}(t) = \delta(t - t_0) = \lim_{\Delta \to 0} f_\Delta(t)$$

This is NOT a function, but one can deal with it as with a function. For example, if $a(t)$ is a continuous function then

$$\int a(t)\delta_{t_0} \, dt = \lim_{\Delta \to 0} \int f_\Delta(t)a(t) \, dt = a(t_0).$$

Its Laplace transform

$$(\mathcal{L}\delta_{t_0})(s) = e^{-st_0}$$

Special term: "distributions" or "generalized functions".
Main idea: *business as usual*

**Example** (old exam problem):

\[
y''(t) + 100y(t) = \delta(t - 2), \quad y(0) = 0, \quad y'(0) = 0.
\]

Test question: can you find \( y(1) \) without solving the equation?

\[
\mathcal{L}: y \mapsto Y(s), \quad y'' \mapsto s^2 Y(s), \quad \delta(t - 2) \mapsto e^{-2s}
\]

\[
Y(s)(s^2 + 100) = e^{-2s}; \quad Y(s) = \frac{e^{-2s}}{s^2 + 100}.
\]

\[
y(t) = \mathcal{L}^{-1} \left( \frac{e^{-2s}}{s^2 + 100} \right) = \frac{1}{10} u(t - 2) \sin 10(t - 2).
\]
Example (almost old example problem)

\[ y''(t) + 100y(t) = \delta(t - 2), \quad y(0) = 0, \quad y(1) = 2. \]
6.5. Convolution

$f, g$ piece-wise continuous functions on $[0, +\infty)$, New function

$$h(t) = (f * g)(t) = \int_0^t f(t - \tau)g(\tau)d\tau = \int_0^t f(\tau)g(t - \tau)d\tau$$

is called the convolution of $f$ and $g$.

Model: river pollution
Properties of the convolution

Basic rules for convolution:

▶ \( f \ast (g_1 + g_2) = f \ast g_1 + f \ast g_2 \)
▶ \( f \ast g = g \ast f \)
▶ \( (f \ast g) \ast w = f \ast (g \ast w) \)

Test question: Do you recognize these properties?
Theorem

Suppose that $f$ and $g$ are piece-wise continuous functions and their Laplace transforms are defined when $s > a$, $\mathcal{L}\{f\} = F$, $\mathcal{L}\{g\} = G$. Then the Laplace transform of their convolution $f \ast g$ is also defined when $s > a$ and

$$\mathcal{L}\{f \ast g\}(s) = F(s)G(s)$$
Convolution: examples

1. \( f(t) = t, \ g(t) = t^2, \ f \ast g = \int_0^t (t - \tau) \tau^2 d\tau = t^4/12 \)
   Check the convolution theorem
   \( \mathcal{L}\{t\} = s^{-2}, \ \mathcal{L}\{t^2\} = 2s^{-3} \) and \( \mathcal{L}\{t^4/12\} = (24s^{-5}) = 2s^{-5}. \)

2. \( f(t) = \cos t, \ g(t) = 1, \ f \ast g = \int_0^t \cos \tau \, d\tau = \sin t \)
   The Laplace transforms are:
   \( \mathcal{L}\{\cos t\} = \frac{s}{s^2+1}, \ \mathcal{L}\{1\} = \frac{1}{s} \) and \( \mathcal{L}\{\sin t\} = \frac{1}{s^2+1} \)

3. \( g(t) = 1, \ f \ast g = \int_0^t f(\tau)d\tau \) and
   \[
   \mathcal{L} \left\{ \int_0^t f(\tau)d\tau \right\} (s) = \frac{1}{s} \mathcal{L}\{f\}(s)
   \]