6.2/6.3 Laplace transform, derivatives, integrals, ODE

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Remind the definition:

$$F(s) = \mathcal{L}f(s) = \int_0^\infty f(t)e^{-st}dt$$

Capital letters will always denote the Laplace transforms of functions denoted by the corresponding small letters

Rules for derivatives:

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0),$$

$$\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0),$$

$$\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$$

Assuming all derivatives exist and grow at most exponentially

Proof for the first derivative

Just integration by parts:

$$(\mathcal{L}f')(s) = \int_0^\infty f'(t)e^{-st}dt =$$

$$\left(f(t)e^{-st}\right)\Big|_0^\infty - \int_0^\infty f(t)\left(e^{-st}\right)' dt =$$

$$s\int_0^\infty f(t)e^{-st}dt - f(0).$$

Exercise: Prove yourself for the second derivative.

Rules for integrals :

$$\mathcal{L}\left(\int_0^t f(\tau)d\tau\right)(s) = \frac{1}{s}F(s).$$

Proof:

Denote: $g(t) = \int_0^t f(\tau)d\tau$. We have $g'(t) = f(t)$ and $g(0) = 0$. Then

$$F(s) = \mathcal{L}(g')(s) = sG(s) - g(0) = sG(s).$$

This is what we need

Example, inverse Laplace transform :

$$\text{Find} \quad \mathcal{L}^{-1}\left(\frac{1}{s(s^2+4)}\right)$$

Solution: Table formula: $\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2} \quad \Rightarrow$

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+4)}\right)(t) = \frac{1}{2}\sin 2t.$$

Integral rule:

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+4)}\right) = \frac{1}{2}\int_0^t \sin 2\tau d\tau = \frac{1}{4} - \frac{1}{4}\cos 2t.$$

Applications to ODE

Solve the initial value problem

$$ay'' + by' + cy = g, \quad y(0) = K_0,\ y'(0) = K_1$$

Apply the Laplace transform

$$(as^2 + bs + c)Y - (as + b)K_0 - aK_1 = G$$

Then

$$Y(s) = \frac{G(s)}{as^2 + bs + c} + \frac{(as + b)K_0 + aK_1}{as^2 + bs + c}$$

We can find $\mathcal{L}\{(as^2 + bs + c)^{-1}\}$ and use the inverse Laplace transform to compute y.

Physical interpretation

$$Y(s) = \underbrace{\frac{G(s)}{as^2 + bs + c}}_{\textit{contribution of the right handside}} + \underbrace{\frac{(as+b)K_0 + aK_1}{as^2 + bs + c}}_{\textit{contribution of the initial conditions}}$$

Transfer function

$$Q(s) = \frac{1}{as^2 + bs + c}$$

Example 1

$$y'' - 9y = 1, \; y(0) = 1, \; y'(0) = 0$$

- Apply the Laplace transform $s^2 Y - s - 9Y = \frac{1}{s}$
- Solve for Y and use partial fractions to write down the answer

$$Y(s) = \frac{s^2+1}{s(s^2-9)} = -\frac{1}{9s} + \frac{5}{9(s-3)} + \frac{5}{9(s+3)}$$

- Find y by performing the inverse transform $y(t) = -1/9 + 5/9e^{3t} + 5/9e^{-3t}$

Example 2

$$y'' + y' - 2y = \sin t, \;\; y(0) = 0, y'(0) = 1$$

1.
$$s^2 Y - 1 + sY - 2Y = \frac{1}{s^2+1}$$

2.
$$Y(s) = \frac{s^2+2}{(s^2+1)(s^2+s-2)}$$

We want to decompose it using partial fractions (see next slide):

$$Y(s) = -0.1\frac{s}{s^2+1} - 0.3\frac{1}{s^2+1} + 0.5\frac{1}{s-1} - 0.4\frac{1}{s+2}$$

3. Applying the inverse transform we get

$$y(t) = -0.1\cos t - 0.3\sin t + 0.5e^{t} - 0.4e^{-2t}$$

Partial fractions: example

We look for the representation

$$Y(s) = \frac{as+b}{s^2+1} + \frac{c}{s-1} + \frac{d}{s+2},$$

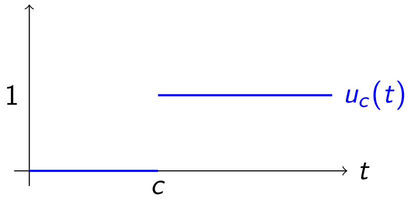
where
$s^2+2 = (as+b)(s-1)(s+2) + c(s^2+1)(s+2) + d(s^2+1)(s-1)$.
Now we either open the brackets, compare the coefficients and solve a system of linear equations or we substitute $s = 1, 2, i$ and find the constants $a = -0.1$, $b = -0.3$, $c = 0.5$, $d = -0.4$.

Exercise: Find $\mathcal{L}^{-1}\left(\frac{1}{s(s^2+4)}\right)$ by applying partial fraction expansion.

Laplace transform of discontinuous functions

The building block for discontinuous functions is the step function (Heaviside's function) u_c:

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$



For $c \geq 0$ we compute its Laplace transform:

$$\mathcal{L}\{u_c\}(s) = \int_c^\infty e^{-st}dt = \frac{e^{-cs}}{s}, \; s > 0.$$

Further, the change of variables gives the second shift rule

$$\mathcal{L}\{u_c(t)f(t-c)\}(s) = e^{-cs}\mathcal{L}\{f\}(s), \quad c \geq 0.$$

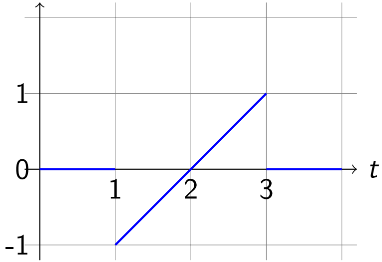
We use it to evaluate the Laplace transform of piecewise defined functions.

Example

Find the Laplace transform of the function

$$f(t) = \begin{cases} 0, & t < 1 \\ t-2, & 1 \le t \le 3 \\ 0, & t > 3 \end{cases}$$

First we look at the graph of this function



And rewrite the function as

$$\begin{aligned} f(t) = &\ (t-2)(u_3(t) - u_1(t)) \\ = &\ (t-3)u_3(t) + u_3(t) \\ &-(t-1)u_1(t) + u_1(t) \end{aligned}$$

Then, using the rules above, we compute

$$\mathcal{L}f(s) = \frac{e^{-3s}}{s^2} + \frac{e^{-3s}}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s}$$

Inverse Laplace transform: Example

An important step in the application of the Laplace transform to ODE is to find the inverse Laplace transform of the given function. Find $f(t)$ such that $\mathcal{L}\{f\} = F$ is

$$F(s) = \frac{e^{-2s}}{s^2 + 2s - 3}$$

First, using the partial functions

$$\frac{1}{s^2 + 2s - 3} = \frac{1}{4}\left(\frac{1}{s-1} - \frac{1}{s+3}\right).$$

Then we write

$$F(s) = \frac{1}{4}\left(\frac{e^{-2s}}{s-1} - \frac{e^{-2s}}{s+3}\right)$$

and using the second shift rule and the table to get

$$\mathcal{L}^{-1}(F)(t) = \frac{u_2(t)}{4}(e^{t-2} - e^{-3(t-2)})$$