6.1 Laplace transform, introduction

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Laplace transform, intro

- Invented by engineers
- Deals with the processes which start at some moment at time (or at space) and extend to infinity.
  Examples:
  - Electrical current, starting at the moment zero;
  - Oscillating mass-spring system
  - More complicated systems

Laplace transform: definition

A special machine which changes one function into another.
Input: $f(t)$, $t > 0$.

Output:

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt$$

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Remind:

$$\int_0^{\infty} f(t)e^{-st}dt = \lim_{A\to\infty} \int_0^{A} f(t)e^{-st}dt$$

if the limit exists.

Laplace transform, first examples

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Example 2 $f(t) = e^{at}$

$$F(s) = \int_0^\infty e^{at} e^{-st} dt = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}, \ s > a.$$

Further examples of the Laplace transform

The Laplace transforms of the following functions can be evaluated using the definition (and integration by parts sometimes)

$f(t)$	$F(s)$
1	$\frac{1}{s}, \; s > 0$
t^n	$\frac{n!}{s^{n+1}}, \; s > 0$
e^{at}	$\frac{1}{s-a}, \; s > a$
$\cos at$	$\frac{s}{s^2+a^2}, \; s > 0$
$\sin at$	$\frac{a}{s^2+a^2}, \; s > 0$
$u_c(t), \; c > 0$	$\frac{e^{-cs}}{s}, \; s > 0$

Laplace transform, piece-wise continuous functions

In applications we have to handle discontinuous external forces (electrical switch, impulse). The mathematical approximation is *piecewise continuous functions*

Definition

A function f is said to be piecewise continuous on an interval $[a, b]$ if this interval can be partitioned into a finite number of intervals such that on each small open interval f is continuous and f has finite one-sided limits on the ends of these sub-intervals.

Example

The Heaviside function

$$u_0(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \qquad u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

Integration of piece-wise continuous functions and improper integrals

If f is a piece-wise continuous function on a finite interval $[a, b]$ then the integral $\int_a^b f(t)dt$ is defined as:

$$\int_a^b f(t)dt = \int_a^{t_1} f(t)dt + \int_{t_1}^{t_2} f(t)dt + ... + \int_{t_n}^b f(t)dt$$

it is the sum of integrals over sub-intervals on which f is continuous.

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Now suppose that f is piece-wise continuous on $[a, A]$ for any $A > a$. Then we consider

$$\int_a^\infty f(t)dt = \lim_{A\to\infty} \int_a^A f(t)dt$$

if the limit exists (we say that the integral converges).

Examples and a comparison theorem

Example

Divergent integrals:

$$\int_0^\infty e^{at}dt,\ \ a \geq 0, \quad \int_1^\infty t^p dt,\ \ p \geq -1, \quad \int_0^\infty \sin t\, dt$$

Convergent integrals:

$$\int_0^\infty e^{at}dt,\ \ a < 0, \quad \int_1^\infty t^p,\ \ p < -1, \quad \int_0^\infty \frac{\sin t}{t} dt$$

Theorem

If $\int_0^\infty g(t)dt$ converges and $|f(t)| < g(t)$ then $\int_0^\infty f(t)dt$ also converges.
If $f(t) > g(t) > 0$ and $\int_0^\infty g(t)dt$ diverges then $\int_0^\infty f(t)dt$ diverges.

Laplace transform: existence

f is a piece-wise continuous function on each interval $[0, A]$;

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt = \lim_{A\to\infty} \int_0^A f(t)e^{-st}dt,$$

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Theorem

If $|f(t)| < M\ e^{kt}$ (such f is called a function of exponential order), then $\mathcal{L}\{f\}(s)$ exists for all $s > k$.

Laplace transform, basic rules

- Linearity
$$\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}$$
- First shift rule
$$\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$$
- Second shift rule
$$\mathcal{L}\{u_c(t)f(t-c)\}(s) = e^{-cs}\mathcal{L}\{f\}(s)$$
- Derivatives
$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0),\ \mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0),$$
$$\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$

Laplace transform, shift theorems

Bla-bla

Example 3, Laplace transform from definition

Find the Laplace transform of the function

$$f(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ t, & 1 \leq t < \infty \end{cases}$$

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Then

$$F(s) = \frac{1}{s} + \frac{e^{-s}}{s^2}, \quad s > 0$$

Example 4,Laplace transform from s-shift

Compute the Laplace transform of the function $e^{at}\sin bt$ if we know that $\mathcal{L}(\sin bt)(s) = \frac{b}{s^2+b^2}$.

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Compute the Laplace transform of the function $e^{at} \sin bt$ if we know that $\mathcal{L}(\sin bt)(s) = \frac{b}{s^2+b^2}$.
We have $\mathcal{L}(e^{at} \sin(bt))(s) = \mathcal{L}(\sin(bt))(s-a) = \frac{b}{(s-a)^2+b^2}$

Laplace transform and derivatives

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0),\ \mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0),$$

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Applications to ODE

Consider an initial value problem for a linear ODE with constant coefficients

$$y^{(n)} + a_{n-1}y^{n-1} + ... + a_1 y' + y = f$$

$$y(0) = K_0, y'(0) = K_1, ..., y^{(n-1)}(0) = K_{n-1}$$

It can be solved by the following procedure:

- apply the Laplace transform to obtain an algebraic equation on $Y = \mathcal{L}\{y\}$
- solve this equation and find Y
- find y such that $\mathcal{L}\{y\} = Y$ (inverse Laplace transform)

Inverse Laplace transform

Definition

Given $Y(s)$ we say that $y(t)$ is the inverse Laplace transform if

$$(\mathcal{L}y)(s) = Y(s)$$

Finding the inverse Laplace transform may be non-trivial !

Example 5: Find the inverse Laplace transform of the function

$$F(s) = \frac{2s}{s^2 - 2s - 3}$$

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We will use the linearity property and table functions. We want to simplify the function $F(s)$ using the partial fraction decomposition:

$$F(s) = \frac{2s}{s^2 - 2s - 3} = \frac{a}{s-3} + \frac{b}{s-1}$$

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Or $2s = a(s-1) + b(s-3)$ and $a = 3, b = -1$.

$$F(s) = \frac{3}{s-3} - \frac{1}{s-1}$$

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Then $f(t) = \mathcal{L}^{-1}(F)(t) = 3e^{3t} - e^t$.