TMA4220 Calculus 4K, Practice problems, Fall 2016

Complex analysis, Residue calculus.

Problem 1. Let $f(z) = z(z + \bar{z})$.
a) Write $f(z)$ in the form $u(x, y)+iv(x, y)$ where $z = x+iy$ and u and v are real-valued functions.
b) Find all point on the complex plane where f is differentiable and compute f' at those points.
c) For each of the functions $u(x, y)$ and $v(x, y)$ determine if it is harmonic (on the whole complex plane) or not. Justify your answer.

Problem 2. Let $g(z) = \overline{f(\bar{z})}$.
a) If $f(z) = e^{(1+i)z}$, find $g(z)$.
b) Suppose that $f(x+iy)) = u(x, y)+iv(x, y)$, where u and v are real-valued. Find the real and imaginary parts of $g(x + iy)$. Show that if $f(z)$ is analytic at z_0 then $g(z)$ is analytic at $\bar{z}_0$.

Problem 3. Let $u(x, y) = x^3 + axy^2$. Determine the value of a for which u is harmonic and for this a find a harmonic function $v(x, y)$ such that $f(x + iy) = u(x, y) + iv(x, y)$ is analytic.

Problem 4. a) Find all complex numbers z such that $e^z = -1$.
b) Show that $|e^z| \le e^{|z|}$ for any complex number z. For which z there is equality?

Problem 5. Find the image of the region $\{z : \mathrm{Re}(z) > 0\}$ under the mapping $w = \mathrm{Ln} z$.

Problem 6. Let $f(z) = \frac{z+2}{z^3-1}$ and C_R be the circle $\{|z| = R\}$.
a) Find all singular points and residues of the function $f(z)$. Compute the integral $\int_{C_2} f(z)dz$ using residues.
b) Use the LM-inequality to show that

$$\left| \int_{C_R} f(z)dz \right| \le \frac{2\pi R(R+2)}{R^3-1}.$$

Let $R \to \infty$ and show that $\int_{C_2} f(z)dz = 0$.

Problem 7. Let $f(z) = \frac{z}{z^2+1}$.
a) Find the Taylor series of $f(z) = \frac{z}{z^2+1}$ at $z_0 = 0$.
b) Find the Taylor series of the function $f'(z)$ at $z_0 = 0$. What is the radius of convergence of this series?
c) Find all Laurent series of f centered at $z_1 = 1$.

Problem 8. Let $f(z) = \frac{z}{1-e^z}$.
a) Determine all singular points of f and classify them (removable singularities, poles, essential singularities).
b) Let $C = \{|z - 1| = 8\}$. Compute $\oint_C f(z)dz$

Problem 9. Let C be a simple closed curve on the complex plane and

$$g(a) = \oint_C \frac{z^3 + z^2 - 3}{(z-a)^3} dz.$$

Show that $g(z) = 0$ when a is outside C and $g(a) = 2\pi i(3a + 1)$ when a is inside C.

Problem 10. Evaluate the following integrals using residues:

$$a) \int_0^\pi \frac{d\theta}{(2+\cos\theta)^2}, \quad b) \int_{-\infty}^{\infty} \frac{x^4}{x^6+4} dx, \quad c) \int_0^\infty \frac{x \sin x}{x^4+1} dx.$$