Problem 1

a) Find the inverse Laplace transform of

\[ F(s) = \frac{s(s + 2)}{s^3 + s^2 + s + 1}. \]

(Hint: \( s^3 + s^2 + s + 1 = (s^2 + 1)(s + 1) \).)

Solution The partial fraction decomposition is

\[ F(s) = \frac{s(s + 2)}{s^3 + s^2 + s + 1} = \frac{3}{2} \frac{s}{s^2 + 1} + \frac{1}{2} \frac{1}{s^2 + 1} - \frac{1}{2} \frac{1}{s + 1} \]

Using the table of the Laplace transforms we obtain

\[ \mathcal{L}^{-1}(F) = \frac{3}{2} \cos t + \frac{1}{2} \sin t - \frac{1}{2} e^{-t} \]

b) Solve the integral equation \( f(t) = \cos t + e^{-2t} \int_0^t f(\tau) e^{2\tau} d\tau \).

Solution We rewrite it as a convolution equation: \( f(t) = \cos t + (f * e^{-2t})(t) \) and apply the Laplace transform. We get

\[ F(s) = \frac{s}{s^2 + 1} + F(s) \frac{1}{s + 2} \]

Solving for \( F \), obtain \( F(s) = \frac{s(s + 2)}{(s^2 + 1)(s + 1)} \). From part a) we know that

\[ f(t) = \frac{3}{2} \cos t + \frac{1}{2} \sin t - \frac{1}{2} e^{-t} \]

Problem 2 Let \( f(x) \) be the 2-periodic function such that \( f(x) = 1 - |x| \) for \( |x| < 1 \).

a) Find the Fourier series of \( f(x) \).

Solution We see that \( f \) is an even function, the Fourier series contains only cosine terms. We compute the coefficients:

\[ a_0 = \frac{1}{2} \int_{-1}^{1} (1 - |x|) dx = \int_{0}^{1} (1 - x) dx = \frac{1}{2}, \]
\[ a_n = \int_{-1}^{1} (1 - |x|) \cos n\pi x \, dx = 2 \int_{0}^{1} (1 - x) \cos n\pi x \, dx = \]
\[ \frac{2}{n\pi} \int_{0}^{1} \sin n\pi x \, dx = \frac{2(1 - \cos n\pi)}{n^2 \pi^2} = \begin{cases} \frac{4}{(2k+1)^2 \pi^2}, & n = 2k + 1 \\ 0, & n = 2k \end{cases} \]

Then
\[ f(x) = \frac{1}{2} + \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{\cos(2k + 1)\pi x}{(2k + 1)^2} \]

b) Find a particular solution of the differential equation \( y'' + 9y = f(x) \).

Solution To find a particular solution, we consider the Fourier series of \( f(x) \) and solve the equation for each term of this series. First we look for a function \( y_0 \) such that \( y_0'' + 9y_0 = 1/2 \), we get \( y_0 = \frac{1}{18} \). Then for each \( n \geq 1 \) we find \( y_n \) such that \( y_n'' + 9y_n = a_n \cos n\pi x \). By the method of undetermined coefficients we look for a solution \( y_n \) of the form
\[ y_n(x) = A_n \cos n\pi x + B_n \sin n\pi x \]

We obtain \( (9 - n^2 \pi^2)A_n = a_n \) and \( B_n = 0 \). Thus \( y_n = \frac{a_n}{9 - n^2 \pi^2} \cos n\pi x \) and summing all terms up,
\[ y(x) = \frac{1}{18} - \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{\cos(2k + 1)x}{(2k + 1)^2((2k + 1)^2 \pi^2 - 9)} \]

Problem 3 Compute the Fourier transform of the function
\[ f(x) = \begin{cases} e^{-|x|} - e^{-1}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \]
and write down the solution of the initial value problem for the heat equation \( u_t = u_{xx} \) for \( -\infty < x < \infty, \ t > 0 \ u(x,0) = f(x) \) in integral form.

Solution We compute the Fourier transform by definition.
\[ \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (e^{-|x|} - e^{-1})e^{-iwx} \, dx \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{-1}^{0} (e^x - e^{-1})e^{-iwx} \, dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{1} (e^{-x} - e^{-1})e^{-iwx} \, dx \]
\[ = \frac{1}{\sqrt{2\pi}} \left( \frac{1 - e^{-(1-iw)}}{1 - iw} - ie^{-1} \frac{1 - e^{iw}}{w} + \frac{1 - e^{-(1+iw)}}{1 + iw} - ie^{-iw} - 1 \right) \]
\[ = \sqrt{\frac{2}{\pi}} \left( \frac{1}{1+w^2} - e^{-1} \frac{\cos w}{1+w^2} - e^{-1} \frac{\sin w}{w(1+w^2)} \right) \]
To write the solution of the heat equation in the integral form we first apply the Fourier transform: \( \hat{u}_t(w, t) = -w^2 \hat{u}(w, t) \) and \( \hat{u}(w, t) = e^{-w^2t} \hat{u}(w, 0) \) we have

\[
    u(x, t) = \mathcal{F}^{-1}(e^{-w^2t} \hat{f}(w))
\]

we can write it either as the inverse Fourier transform:

\[
    u(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-w^2t}}{1 + w^2} (1 - e^{-1} \cos w - e^{-1} \frac{\sin w}{w}) e^{ixw} dw
\]

or better, using the convolution theorem and the formula \( e^{-w^2t} = (2t)^{-1/2} \mathcal{F}(e^{-x^2/(4t)}) \), as

\[
    u(x, t) = \frac{1}{\sqrt{2t}} \mathcal{F}^{-1}(f \cdot e^{-x^2/(4t)}) = \frac{1}{2\sqrt{\pi t}} f * e^{-x^2/(4t)} = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} (e^{-|y|} - e^{-1}) e^{-\frac{(x-y)^2}{4t}} dy
\]

**Problem 4** Find the image of the half-plane \( \{Rez > 0\} \) under the mapping \( w = e^z \).

*Solution* We have \( w = e^z = e^{x+iy} = e^xe^{iy} \). Thus \( |w| = e^x \) and \( arg(w) = y \). The half-plane \( \{Rez > 0\} \) is the set where \( x > 0 \), the image of this set is \( \{w : |w| > 1\} \). The image of this half-plane is the set \( \{w : |w| > 1\} \), it is the exterior of the unit disk.
Problem 5  Consider the series $\sum_{n=1}^{\infty} \frac{3^n}{2n} z^{2n}$.

a) Find the radius of convergence of this series.

*Solution* We can apply the ratio test for this series. Let $a_n = (2n)^{-1} 3^n z^{2n}$, then

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)3|z|^2}{n} \to 3|z|^2, \quad n \to \infty.$$ 

We see that when $|z| < 1/\sqrt{3}$ the series converges and when $|z| > 1/\sqrt{3}$ it diverges. The radius of convergence is $R = 1/\sqrt{3}$.

b) Let $f(z)$ be the sum of the series, write down the series expansion of $f'(z)$ and find $f'(z)$.

*Solution* We differentiate each term and obtain

$$f'(z) = \sum_{n=1}^{\infty} 3^n z^{2n-1} = 3z \sum_{n=1}^{\infty} 3^{n-1} z^{2n-2} = \frac{3z}{1 - 3z}, \quad |z| < 1/\sqrt{3}.$$ 

We have used the formula for the sum of the geometric series in the last equality.

c) Show that $f(z) = -\frac{1}{2} \ln(1 - 3z^2)$ in a disk around the origin.

*Solution* We know that $f'(z) = \frac{3z}{1 - 3z^2}$ and $f(0) = 0$ from the series expansion. Consider the function $g(z) = -\frac{1}{2} \ln(1 - 3z^2)$. Clearly, $g(0) = \ln(1) = 0$ and

$$g'(z) = -\frac{1}{2} \frac{-6z}{1 - 3z^2} = \frac{3z}{1 - 3z^2}.$$

Hence $g(z) = f(z)$ when $|z| < 1/\sqrt{3}$.

Problem 6  Evaluate the integral $I = \int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta}.$

*Solution 1* We use complex parametrization $e^{i\theta} = z$, $z$ is on the unit circle, $C = \{z : |z| = 1\}$. Then $\sin \theta = (z - 1/z)/(2i)$, $d\theta = dz/(iz)$

$$I = \int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta} = \oint_{C} \frac{dz}{iz(1 - (z - z^{-1})^2)/4}$$

$$= \oint_{C} \frac{4z^2 dz}{iz(4z^2 - (z^2 - 1)^2)} = 4i \oint_{C} \frac{z dz}{z^4 - 6z^2 + 1}$$
Now the function $f(z) = \frac{z}{z^4 - 4z^2 + 1}$ has four poles, $z_1, z_2, z_3, z_4$ that are zeros of the polynomial $q(z) = z^4 - 6z^2 + 1$. To find the poles we first solve the equation $q(z) = 0$ as an equation in $z^2$. We get $$z^2 = 3 \pm \sqrt{8} = 3 \pm 2\sqrt{2}$$

We get two positive real solutions for $z^2$. Then $$z_1 = \sqrt{3 - 2\sqrt{2}}, \quad z_2 = -\sqrt{3 - 2\sqrt{2}}, \quad z_3 = \sqrt{3 + 2\sqrt{2}}, \quad z_4 = -\sqrt{3 - 2\sqrt{2}}$$

(You may note that $(1 + \sqrt{2})^2 = 3 + 2\sqrt{2}$ and $(1 - \sqrt{2})^2 = 3 - 2\sqrt{2}$ but it is not necessary to finish the computation.)

We see that $|z_3| = |z_4| > 1$ and $|z_1| = |z_2| < 1$. Then the residue theorem implies

$$I = 4i \oint_C f(z) dz = 4i(2\pi i)(\text{Res}_{z_1} f(z) + \text{Res}_{z_2} f(z)) = -8\pi (\text{Res}_{z_1} f(z) + \text{Res}_{z_2} f(z))$$

Now, to compute the residues in simple poles we use the formula $\text{Res}_{z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$.

$$\text{Res}_{z_j} \frac{z}{z^4 - 6z^2 + 1} = \frac{z_j}{4z_j^3 - 12z_j} = \frac{1}{4z_j^2 - 12}$$

For $z_1$ and $z_2$ we have $z_j^2 = 3 \mp 2\sqrt{2}$ and

$$\text{Res}_{z_j} \frac{z}{z^4 - 6z^2 + 1} = \frac{1}{4(-2\sqrt{2})} = -\frac{\sqrt{2}}{16}, \quad j = 1, 2.$$  

Finally, $I = -8\pi \left(-\frac{\sqrt{2}}{8}\right) = \pi \sqrt{2}$.

**Solution 2** We start with the formula $\sin^2 \theta = (1 - \cos 2\theta)/2$. Then

$$I = \int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta} = \int_{-\pi}^{\pi} \frac{2d\theta}{3 - \cos 2\theta} = \int_{-2\pi}^{2\pi} \frac{d\phi}{3 - \cos \phi}$$

We integrate a $2\pi$-periodic function over the interval $(-2\pi, 2\pi)$ it is twice the period. Then

$$I = 2 \int_{-\pi}^{\pi} \frac{d\phi}{3 - \cos \phi}.$$

Now we apply the complex parametrization

$$I = 2 \oint_C \frac{dz}{iz(3 - (z + z^{-1})/2)} = 4 \oint_C \frac{dz}{i(6z - z^2 - 1)} = 4i \oint_C \frac{dz}{z^2 - 6z + 1}.$$  

The function $g(z) = (z^2 - 6z + 1)^{-1}$ has two simple poles $w_1 = 3 - 2\sqrt{2}$ and $w_2 = 3 + 2\sqrt{2}$. Only the first one is inside the unit disk. Then the residue theorem gives

$$I = -8\pi \text{Res}_{w_1} g(z) = -8\pi \frac{1}{2w_1 - 6} = \frac{8\pi}{4\sqrt{2}} = \pi \sqrt{2}.$$