Problem 1

a) Find the inverse Laplace transform of

$$F(s) = \frac{s(s+2)}{s^3+s^2+s+1}.$$

(Hint: $s^3+s^2+s+1 = (s^2+1)(s+1)$.)

Solution The partial fraction decomposition is

$$F(s) = \frac{s(s+2)}{s^3+s^2+s+1} = \frac{3}{2}\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s^2+1} - \frac{1}{2}\frac{1}{s+1}$$

Using the table ofthe Laplace transforms we obtain

$$\underline{\mathcal{L}^{-1}(F) = \frac{3}{2}\cos t + \frac{1}{2}\sin t - \frac{1}{2}e^{-t}}$$

b) Solve the integral equation $f(t) = \cos t + e^{-2t}\int_0^t f(\tau)e^{2\tau}d\tau$.

Solution We rewrite it as a convolution equation: $f(t) = \cos t + (f * e^{-2t})(t)$ and apply the Laplace transform. We get

$$F(s) = \frac{s}{s^2+1} + F(s)\frac{1}{s+2}$$

Solving for F,obtain $F(s) = \frac{s(s+2)}{(s^2+1)(s+1)}$.From part a) we know that

$$\underline{f(t) = \frac{3}{2}\cos t + \frac{1}{2}\sin t - \frac{1}{2}e^{-t}}$$

Problem 2 Let $f(x)$ be the 2-periodic function such that $f(x) = 1 - |x|$ for $|x| < 1$.

a) Find the Fourier series of $f(x)$.

Solution We see that f is an even function,the Fourier series contains only cosine terms. We compute the coefficients:

$$a_0 = \frac{1}{2}\int_{-1}^{1}(1-|x|)dx = \int_0^1(1-x)dx = \frac{1}{2},$$

$$a_n = \int_{-1}^{1} (1-|x|)\cos n\pi x dx = 2\int_0^1 (1-x)\cos n\pi x dx =$$

$$\frac{2}{n\pi}\int_0^1 \sin n\pi x dx = \frac{2(1-\cos n\pi)}{n^2\pi^2} = \begin{cases} \frac{4}{(2k+1)^2\pi^2}, & n = 2k+1 \\ 0, & n = 2k \end{cases}$$

Then

$$\underline{f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{k=0}^{\infty} \frac{\cos(2k+1)\pi x}{(2k+1)^2}}$$

b) Find a particular solution of the differential equation $y'' + 9y = f(x)$.

Solution To find a particular solution, we consider the Fourier series of $f(x)$ and solve the equation for each term of this series. First we look for a function y_0 such that $y_0'' + 9y_0 = 1/2$, we get $y_0 = \frac{1}{18}$. Then for each $n \geq 1$ we find y_n such that $y_n'' + 9y_n = a_n \cos n\pi x$. By the method of undetermined coefficients we look for a solution y_n of the form

$$y_n(x) = A_n \cos n\pi x + B_n \sin n\pi x$$

We obtain $(9 - n^2\pi^2)A_n = a_n$ and $B_n = 0$. Thus $y_n = \frac{a_n}{9-n^2\pi^2}\cos n\pi x$ and summing all terms up,

$$\underline{y(x) = \frac{1}{18} - \frac{4}{\pi^2}\sum_{k=0}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2((2k+1)^2\pi^2 - 9)}}$$

Problem 3 Compute the Fourier transform of the function

$$f(x) = \begin{cases} e^{-|x|} - e^{-1}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

and write down the solution of the initial value problem for the heat equation $u_t = u_{xx}$ for $-\infty < x < \infty,\ t > 0$ $u(x,0) = f(x)$ in integral form.

Solution We compute the Fourier transform by definition.

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} (e^{-|x|} - e^{-1})e^{-iwx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{0} (e^{x} - e^{-1})e^{-iwx}dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{1} (e^{-x} - e^{-1})e^{-iwx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-(1-iw)}}{1-iw} - ie^{-1}\frac{1-e^{iw}}{w} + \frac{1-e^{-(1+iw)}}{1+iw} - ie^{-1}\frac{e^{-iw}-1}{w}\right)$$

$$\underline{= \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+w^2} - e^{-1}\frac{\cos w}{1+w^2} - e^{-1}\frac{\sin w}{w(1+w^2)}\right)}$$

To write the solution of the heat equation in the integral form we first apply the Fourier transform: $\hat{u}_t(w,t) = -w^2\hat{u}(w,t)$ and $\hat{u}(w,t) = e^{-w^2t}\hat{u}(w,0)$ we have

$$u(x,t) = \mathcal{F}^{-1}(e^{-w^2t}\hat{f}(w))$$

we can write it either as the inverse Fourier transform:

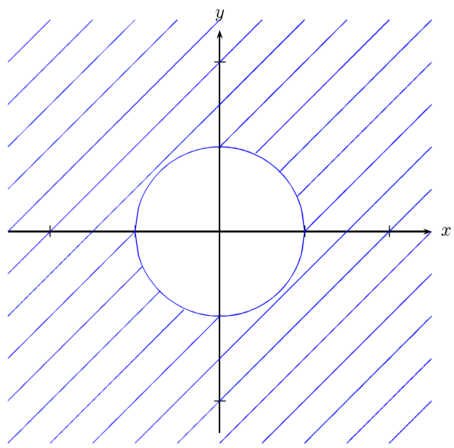
$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-w^2t}}{1+w^2}(1 - e^{-1}\cos w - e^{-1}\frac{\sin w}{w})e^{ixw}dw$$

or better, using the convolution theorem and the formula $e^{-w^2t} = (2t)^{-1/2}\mathcal{F}(e^{-x^2/(4t)})$, as

$$u(x,t) = \frac{1}{\sqrt{2t}}\mathcal{F}^{-1}(\hat{f}\widehat{e^{-x^2/(4t)}}) = \frac{1}{2\sqrt{\pi t}}f * e^{-x^2/(4t)} = \underline{\frac{1}{2\sqrt{\pi t}}\int_{-1}^{1}(e^{-|y|} - e^{-1})e^{-\frac{(x-y)^2}{4t}}dy}$$

Problem 4 Find the image of the half-plane $\{\mathrm{Re} z > 0\}$ under the mapping $w = e^z$.

Solution We have $w = e^z = e^{x+iy} = e^x e^{iy}$. Thus $|w| = e^x$ and $arg(w) = y$. The half-plane $\{\mathrm{Re} z > 0\}$ is the set where $x > 0$, the image of this set is $\{w : |w| > 1\}$. The image of this half-plane is the set $\{w : |w| > 1\}$, it is the exterior of the unit disk.



Problem 5 Consider the series $\sum_{n=1}^{\infty} \frac{3^n}{2n} z^{2n}$.

a) Find the radius of convergence of this series.

Solution We can apply the ratio test for this series. Let $a_n = (2n)^{-1} 3^n z^{2n}$, then

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)3|z|^2}{n} \to 3|z|^2, \quad n \to \infty.$$

We see that when $z| < 1/\sqrt{3}$the series converges and when $|z| > 1/\sqrt{3}$ it diverges. The radius of convergence is $\underline{R = 1/\sqrt{3}}$.

b) Let $f(z)$ be the sum of the series, write down the series expansion of $f'(z)$ and find $f'(z)$.

Solution We differentiate each term and obtain

$$f'(z) = \underline{\sum_{n=1}^{\infty} 3^n z^{2n-1}} = 3z \sum_{n=1}^{\infty} 3^{n-1} z^{2n-2} = \underline{\frac{3z}{1-3z}}, \quad |z| < 1/\sqrt{3}$$

We have used the formula for the sum of the geometric series in the last equality.

c) Show that $f(z) = -\frac{1}{2}\text{Ln}(1-3z^2)$ in a disk around the origin.

Solution We know that $f'(z) = \frac{3z}{1-3z^2}$ and $f(0) = 0$ from the series expansion. Consider the function $g(z) = -\frac{1}{2}\text{Ln}(1-3z^2)$. Clearly, $g(0) = \text{Ln}(1) = 0$ and

$$g'(z) = -\frac{1}{2}\frac{-6z}{1-3z^2} = \frac{3z}{1-3z^2}.$$

Hence $g(z) = f(z)$ when $|z| < 1/\sqrt{3}$.

Problem 6 Evaluate the integral $\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}$.

Solution 1 We use complex parametrization $e^{i\theta} = z$, z is on the unit circle, $C = \{z : |z| = 1\}$. Then $\sin\theta = (z - 1/z)/(2i)$, $d\theta = dz/(iz)$

$$I = \int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \oint_C \frac{dz}{iz(1-(z-z^{-1})^2)/4)}$$
$$= \oint_C \frac{4z^2 dz}{iz(4z^2-(z^2-1)^2)} = 4i \oint_C \frac{z dz}{z^4 - 6z^2 + 1}$$

Now the function $f(z) = \frac{z}{z^4-6z^2+1}$ has four poles, z_1, z_2,z_3,z_4 that are zeros of the polynomial $q(z) = z^4 - 6z^2 + 1$. To find the poles we first solve the equation $q(z) = 0$as an equation in z^2. We get

$$z^2 = 3 \pm \sqrt{8} = 3 \pm 2\sqrt{2}$$

We get two positive real solutions for z^2. Then

$$z_1 = \sqrt{3-2\sqrt{2}}, \quad z_2 = -\sqrt{3-2\sqrt{2}}, \quad z_3 = \sqrt{3+2\sqrt{2}}, \quad z_4 = -\sqrt{3-2\sqrt{2}}$$

(You may note that $(1+\sqrt{2})^2 = 3+2\sqrt{2}$ and $(1-\sqrt{2})^2 = 3-2\sqrt{2}$ but it is not necessary to finish the computation.)

We see that $|z_3| = |z_4| > 1$ and $|z_1| = |z_2| < 1$. Then the residue theorem implies

$$I = 4i \oint_C f(z)dz = 4i(2\pi i)(\mathrm{Res}_{z_1} f(z) + \mathrm{Res}_{z_2} f(z)) = -8\pi(\mathrm{Res}_{z_1} f(z) + \mathrm{Res}_{z_2} f(z))$$

Now, to compute the residues in simple poles we use the formula $\mathrm{Res}_{z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$.

$$\mathrm{Res}_{z_j} \frac{z}{z^4 - 6z^2 + 1} = \frac{z_j}{4z_j^3 - 12z_j} = \frac{1}{4z_j^2 - 12}$$

For $z_1$and $z_2$we have $z_j^2 = 3 - 2\sqrt{2}$ and

$$\mathrm{Res}_{z_j} \frac{z}{z^4 - 6z^2 + 1} = \frac{1}{4(-2\sqrt{2}} = -\frac{\sqrt{2}}{16}, \quad j = 1, 2.$$

Finally, $I = -8\pi(-\sqrt{2}/8) = \underline{\pi\sqrt{2}}$.

Solution 2 We start with the formula $\sin^2\theta = (1 - \cos 2\theta)/2$. Then

$$I = \int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2\theta} = \int_{-\pi}^{\pi} \frac{2d\theta}{3 - \cos 2\theta} = \int_{-2\pi}^{2} \pi \frac{d\phi}{3 - \cos\phi}$$

We integrate a 2π-periodic function over the interval $(-2\pi, 2\pi)$it is twice the period. Then

$$I = 2\int_{-\pi}^{\pi} \frac{d\phi}{3 - \cos\phi}.$$

Now we apply the complex parametrization

$$I = 2\oint_C \frac{dz}{iz(3 - (z + z^{-1})/2)} = 4\oint_C \frac{dz}{i(6z - z^2 - 1)} = 4i\oint_C \frac{dz}{z^2 - 6z + 1}.$$

The function $g(z) = (z^2 - 6z + 1)^{-1}$ has two simple poles $w_1 = 3 - 2\sqrt{2}$ and $w_2 = 3 + 2\sqrt{2}$. Only the first one is inside the unit disk. Then the residue theorem gives

$$I = -8\pi \mathrm{Res}_{w_1} g(z) = -8\pi \frac{1}{2w_1 - 6} = \frac{8\pi}{4\sqrt{2}} = \underline{\pi\sqrt{2}}.$$