Problem 1

a) Find the inverse Laplace transform of

\[ F(s) = \frac{s(s + 2)}{s^3 + s^2 + s + 1}. \]

(Hint: \( s^3 + s^2 + s + 1 = (s^2 + 1)(s + 1) \).)

b) Solve the integral equation \( f(t) = \cos t + e^{-2t} \int_0^t f(\tau) e^{2\tau} d\tau. \)

Problem 2

Let \( f(x) \) be the 2-periodic function such that \( f(x) = 1 - |x| \) for \( |x| < 1 \).

a) Find the Fourier series of \( f(x) \).

b) Find a particular solution of the differential equation \( y'' + 9y = f(x) \).

Problem 3

Compute the Fourier transform of the function

\[ f(x) = \begin{cases} 
  e^{-|x|} - e^{-1}, & |x| < 1 \\
  0, & |x| \geq 1 
\end{cases} \]

and write down the solution of the initial value problem for the heat equation \( u_t = u_{xx} \) for \( -\infty < x < \infty, \ t > 0 \ u(x,0) = f(x) \) in integral form.

Problem 4

Find the image of the half-plane \( \{\Re z > 0\} \) under the mapping \( w = e^z \).

Problem 5

Consider the series \( \sum_{n=1}^{\infty} \frac{3^n}{2n} z^{2n} \).

a) Find the radius of convergence of this series.

b) Let \( f(z) \) be the sum of the series, write down the series expansion of \( f'(z) \) and find \( f'(z) \).

c) Show that \( f(z) = -\frac{1}{2} \ln(1 - 3z^2) \) in a disk around the origin.

Problem 6

Evaluate the integral \( \int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta} \).