PROBLEM SET 11.7

1–6 EVALUATION OF INTEGRALS

Show that the integral represents the indicated function. *Hint.* Use (5), (10), or (11); the integral tells you which one, and its value tells you what function to consider. Show your work in detail.

1. $$\int_0^\infty \frac{\cos xw + w\sin xw}{1+w^2}\,dx = \begin{cases} 0 & \text{if } x<0 \\ \pi/2 & \text{if } x=0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$$

2. $$\int_0^\infty \frac{\sin \pi w \sin xw}{1-w^2}\,dw = \begin{cases} \frac{\pi}{2}\sin x & \text{if } 0 \leqq x \leqq \pi \\ 0 & \text{if } x>\pi \end{cases}$$

3. $$\int_0^\infty \frac{1-\cos \pi w}{w}\sin xw\,dw = \begin{cases} \frac{1}{2}\pi & \text{if } 0<x<\pi \\ 0 & \text{if } x>\pi \end{cases}$$

4. $$\int_0^\infty \frac{\cos \frac{1}{2}\pi w}{1-w^2}\cos xw\,dw = \begin{cases} \frac{1}{2}\pi\cos x & \text{if } 0<|x|<\frac{1}{2}\pi \\ 0 & \text{if } |x| \geqq \frac{1}{2}\pi \end{cases}$$

5. $$\int_0^\infty \frac{\sin w - w\cos w}{w^2}\sin xw\,dw = \begin{cases} \frac{1}{2}\pi x & \text{if } 0<x<1 \\ \frac{1}{4}\pi & \text{if } x=1 \\ 0 & \text{if } x>1 \end{cases}$$

6. $$\int_0^\infty \frac{w^3\sin xw}{w^4+4}\,dw = \tfrac{1}{2}\pi e^{-x}\cos x \quad \text{if } x>0$$

7–12 FOURIER COSINE INTEGRAL REPRESENTATIONS

Represent $f(x)$ as an integral (10).

7. $f(x) = \begin{cases} 1 & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$

8. $f(x) = \begin{cases} x^2 & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$

9. $f(x) = 1/(1+x^2)$ [$x>0$. *Hint.* See (13).]

10. $f(x) = \begin{cases} a^2-x^2 & \text{if } 0<x<a \\ 0 & \text{if } x>a \end{cases}$

11. $f(x) = \begin{cases} \sin x & \text{if } 0<x<\pi \\ 0 & \text{if } x>\pi \end{cases}$

12. $f(x) = \begin{cases} e^{-x} & \text{if } 0<x<a \\ 0 & \text{if } x>a \end{cases}$

13. CAS EXPERIMENT. Approximate Fourier Cosine Integrals. Graph the integrals in Prob. 7, 9, and 11 as functions of x. Graph approximations obtained by replacing ∞ with finite upper limits of your choice. Compare the quality of the approximations. Write a short report on your empirical results and observations.

14. PROJECT. Properties of Fourier Integrals

(a) Fourier cosine integral. Show that (10) implies

(a1) $$f(ax) = \frac{1}{a}\int_0^\infty A\left(\frac{w}{a}\right)\cos xw\,dw$$
$(a>0)$ (*Scale change*)

(a2) $$xf(x) = \int_0^\infty B^*(w)\sin xw\,dw,$$
$$B^* = -\frac{dA}{dw}, \quad A \text{ as in (10)}$$

(a3) $$x^2 f(x) = \int_0^\infty A^*(w)\cos xw\,dw,$$
$$A^* = -\frac{d^2A}{dw^2}.$$

(b) Solve Prob. 8 by applying (a3) to the result of Prob. 7.

(c) Verify (a2) for $f(x)=1$ if $0<x<a$ and $f(x)=0$ if $x>a$.

(d) Fourier sine integral. Find formulas for the Fourier sine integral similar to those in (a).

15. CAS EXPERIMENT. Sine Integral. Plot $\mathrm{Si}(u)$ for positive u. Does the sequence of the maximum and minimum values give the impression that it converges and has the limit $\pi/2$? Investigate the Gibbs phenomenon graphically.

16–20 FOURIER SINE INTEGRAL REPRESENTATIONS

Represent $f(x)$ as an integral (11).

16. $f(x) = \begin{cases} x & \text{if } 0<x<a \\ 0 & \text{if } x>a \end{cases}$

17. $f(x) = \begin{cases} 1 & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$

18. $f(x) = \begin{cases} \cos x & \text{if } 0<x<\pi \\ 0 & \text{if } x>\pi \end{cases}$

19. $f(x) = \begin{cases} e^{x} & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$

20. $f(x) = \begin{cases} e^{-x} & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$

PROBLEM SET 11.9

1. **Review in complex.** Show that $1/i = -i$, $e^{-ix} = \cos x - i \sin x$, $e^{ix} + e^{-ix} = 2 \cos x$, $e^{ix} - e^{-ix} = 2i \sin x$, $e^{ikx} = \cos kx + i \sin kx$.

2–11 FOURIER TRANSFORMS BY INTEGRATION

Find the Fourier transform of $f(x)$ (without using Table III in Sec. 11.10). Show details.

2. $f(x) = \begin{cases} e^{2ix} & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

3. $f(x) = \begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$

4. $f(x) = \begin{cases} e^{kx} & \text{if } x < 0 \quad (k > 0) \\ 0 & \text{if } x > 0 \end{cases}$

5. $f(x) = \begin{cases} e^{x} & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$

6. $f(x) = e^{-|x|} \quad (-\infty < x < \infty)$

7. $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$

8. $f(x) = \begin{cases} xe^{-x} & \text{if } -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}$

9. $f(x) = \begin{cases} |x| & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

10. $f(x) = \begin{cases} x & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

11. $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

12–17 USE OF TABLE III IN SEC. 11.10. OTHER METHODS

12. Find $\mathcal{F}(f(x))$ for $f(x) = xe^{-x}$ if $x > 0, f(x) = 0$ if $x < 0$, by (9) in the text and formula 5 in Table III (with $a = 1$). *Hint.* Consider xe^{-x} and e^{-x}.
13. Obtain $\mathcal{F}(e^{-x^2/2})$ from Table III.
14. In Table III obtain formula 7 from formula 8.
15. In Table III obtain formula 1 from formula 2.
16. **TEAM PROJECT. Shifting (a)** Show that if $f(x)$ has a Fourier transform, so does $f(x - a)$, and $\mathcal{F}\{f(x - a)\} = e^{-iwa}\mathcal{F}\{f(x)\}$.

    (b) Using (a), obtain formula 1 in Table III, Sec. 11.10, from formula 2.

    (c) Shifting on the w-Axis. Show that if $\hat{f}(w)$ is the Fourier transform of $f(x)$, then $\hat{f}(w - a)$ is the Fourier transform of $e^{iax}f(x)$.

    (d) Using (c), obtain formula 7 in Table III from 1 and formula 8 from 2.
17. What could give you the idea to solve Prob. 11 by using the solution of Prob. 9 and formula (9) in the text? Would this work?

18–25 DISCRETE FOURIER TRANSFORM

18. Verify the calculations in Example 4 of the text.
19. Find the transform of a general signal $f = [f_1 \ \ f_2 \ \ f_3 \ \ f_4]^\mathsf{T}$ of four values.
20. Find the inverse matrix in Example 4 of the text and use it to recover the given signal.
21. Find the transform (the frequency spectrum) of a general signal of two values $[f_1 \ \ f_2]^\mathsf{T}$.
22. Recreate the given signal in Prob. 21 from the frequency spectrum obtained.
23. Show that for a signal of eight sample values, $w = e^{-i/4} = (1 - i)/\sqrt{2}$. Check by squaring.
24. Write the Fourier matrix **F** for a sample of eight values explicitly.
25. **CAS Problem.** Calculate the inverse of the 8×8 Fourier matrix. Transform a general sample of eight values and transform it back to the given data.