TMA4120 Matematikk 4K

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Løsningsforslag

1 We apply the Laplace transform to the differential equation $y'' + 2y = 2\sin(t)$ and we get the equation

$$s^2Y - sf(0) - f'(0) + 2Y = 2\frac{1}{s^2+1},$$

where $Y = \mathcal{L}(y)$. Since $f'(0) = f(0) = 0$ we have that

$$s^2Y + 2Y = 2\frac{1}{s^2+1},$$

and isolating the Y we get

$$Y = 2\frac{1}{(s^2+1)(s^2+1)}.$$

Now using fraction reduction we have that

$$Y = \frac{2}{s^2+1} - \frac{2}{s^2+2},$$

and hence applying the inverse of the Laplace transform we get

$$\mathcal{L}^{-1}(Y) = y(t) = 2\mathcal{L}^{-1}(\frac{1}{s^2+1}) - 2\mathcal{L}^{-1}(\frac{1}{s^2+2}) = 2\sin(t) - \frac{2}{\sqrt{2}}\sin(\sqrt{2}t).$$

2 The Fourier series of $f(x) = \cosh(t)$ on the interval $[-\pi, \pi]$ is given by

$$g(x) = \frac{1}{\pi}\sinh(\pi)\left(1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cos(nx)\right).$$

Since $\lim_{x\to\pi^+} g(x) = \lim_{x\to\pi^-} g(x) = \cosh(\pi)$ we have that

$$\cosh(\pi) = \frac{1}{\pi}\sinh(\pi)\left(1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cos(n\pi)\right) = \frac{1}{\pi}\sinh(\pi)\left(1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(-1)^n\right)$$
$$\frac{1}{\pi}\sinh(\pi)\left(1 + 2\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{1+n^2}\right) = \frac{1}{\pi}\sinh(\pi)\left(1 + 2\sum_{n=1}^{\infty}\frac{1}{1+n^2}\right).$$

Therefore,

$$\pi\frac{\cosh(\pi)}{\sinh(\pi)} = 1 + 2\sum_{n=1}^{\infty}\frac{1}{1+n^2},$$

and

$$\sum_{n=1}^{\infty}\frac{1}{1+n^2} = \frac{\pi}{2}\coth(\pi) - \frac{1}{2}.$$

3 (a) Standard separation of variables gives

$$F'' + kF = 0, \quad G'' - kG = 0$$

for some constant $k \in \mathbb{R}$. Three cases to be considered:

(i) $k = -\lambda^2 < 0$. Using the boundary condition $0 = F'(0) = F'(\pi)$ we see that F is identically zero.

(ii) $k = 0$. Using the boundary condition $0 = F'(0) = F'(\pi)$, we see that F constant is the only solution.

(iii) $k = \lambda^2 > 0$. Using the boundary condition $0 = F'(0) = F'(\pi)$, we see in the standard manner that $\lambda = n \in \mathbb{Z}$ and

$$F(x) = \beta_n \cos(nx), \quad n \in \mathbb{Z},$$

for any constant $\beta_n \in \mathbb{R}$. Since cosinus is an even function it suffices to consider nonnegative integers.

Using this result for the equation for G, we infer that

$$G(y) = A_n e^{ny} + B_n e^{-ny}, \quad n \in \mathbb{N},$$

for any constants $A_n, B_n \in \mathbb{R}$. For $n = 0$ we find $G(y) = A_0 y + B_0$ for constants $A_0, B_0 \in \mathbb{R}$.

Thus the general solution of the form $u = FG$ reads

$$u(x,y) = F(x)G(y) = u_n(x,y) = \begin{cases} A_0 y + B_0, & \text{for } n = 0, \\ \cos(nx)\big(A_n e^{ny} + B_n e^{-ny}\big), & \text{for } n \in \mathbb{N}. \end{cases}$$

(b) The general solution reads

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) = A_0 y + B_0 + \sum_{n=1}^{\infty} \cos(nx)\big(A_n e^{ny} + B_n e^{-ny}\big).$$

The boundary condition at $y = 0$ yields

$$u(x,0) = B_0 + \sum_{n=1}^{\infty} \cos(nx)\big(A_n + B_n\big) = 0.$$

By the uniqueness of Fourier series we conclude that $B_0 = 0$ and $A_n = -B_n$ for $n \in \mathbb{N}$. The boundary condition at $y = \frac{\pi}{2}$ yields

$$\begin{aligned} u_y(x, \frac{\pi}{2}) &= A_0 + \sum_{n=1}^{\infty} \cos(nx) n A_n \big(e^{n\pi/2} + e^{-n\pi/2}\big) \\ &= (1 + \cos(x))^2 \\ &= 1 + 2\cos(x) + \cos^2(x) \\ &= \frac{3}{2} + 2\cos(x) + \frac{1}{2}\cos(2x). \end{aligned}$$

Again by the uniqueness of Fourier series we conclude that $A_0 = 3/2$, $1A_1(e^{\pi/2} + e^{-\pi/2}) = 2$, and $2A_2(e^{\pi} + e^{-\pi}) = 1/2$, while all the other constants vanish. We may write

$$A_1 = \frac{1}{\cosh(\pi/2)}, \quad A_2 = \frac{1}{8\cosh(\pi)}.$$

Thus

$$u(x,y) = \frac{3}{2}y + \frac{1}{\cosh(\pi/2)}\cos(x)\big(e^{y} - e^{-y}\big) + \frac{1}{8\cosh(\pi)}\cos(2x)\big(e^{2y} - e^{-2y}\big).$$

4 We take the Fourier transform (Kreyzsig, p. 527)

$$\begin{aligned}
\mathcal{F}\big(f * g\big) &= \sqrt{2\pi}\mathcal{F}(f)\,\mathcal{F}(g) \\
&= \sqrt{2\pi}\,\frac{1}{\sqrt{2}}e^{-\omega^2/4}\big(-\frac{1}{2}\big)\mathcal{F}\left(\frac{d}{dx}e^{-x^2}\right) \\
&= -\frac{\sqrt{2\pi}}{2\sqrt{2}}\,e^{-\omega^2/4}\,i\omega\mathcal{F}\big(e^{-x^2}\big) \\
&= -\frac{\sqrt{2\pi}i}{4}\,\omega\,e^{-\omega^2/4}\,e^{-\omega^2/4} \\
&= -\frac{\sqrt{2\pi}i}{4}\omega\,e^{-\omega^2/2}.
\end{aligned}$$

Here we also used the formula for the Fourier transform of the derivative (Kreyzsig, p. 526).

Using that $h(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\mathcal{F}(h)e^{i\omega x}d\omega$ (Kreyzsig, p. 524) we see that

$$\begin{aligned}
f * g &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\mathcal{F}(f*g)e^{i\omega x}d\omega \\
&= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} -\frac{\sqrt{2\pi}i}{4}\omega\,e^{-\omega^2/2}e^{i\omega x}d\omega \\
&= -\frac{1}{\sqrt{2\pi}}\frac{\sqrt{2\pi}i}{4}\int\limits_{-\infty}^{\infty}\omega\,e^{-\omega^2/2}e^{i\omega x}d\omega \\
&= -\frac{i}{4}\int\limits_{-\infty}^{\infty}\omega\,e^{-\omega^2/2}e^{i\omega x}d\omega
\end{aligned}$$

5 For $z = x + iy \in \mathbb{C}\setminus\{0\}$ we have that

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = (x+iy)^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\,.$$

So we have that $f(z) = u(x,y) + iv(x,y)$ where

$$u(x,y) = \frac{x}{x^2+y^2} \qquad \textbf{and} \qquad v(x,y) = -\frac{y}{x^2+y^2}\,.$$

The Cauchy Riemann equations say that

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x\,.$$

But

$$u_x = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}\,,$$

$$u_y = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{0 - x(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}\,,$$

$$v_x = \frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right) = -\frac{0 - x(2y)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}\,,$$

and

$$v_y = \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = -\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}\,.$$

So the Cauchy-Riemann equations are trivially satisfied.

Finally using Theorem 2 in page 627 Kreyzsig, it follows that $f(x+iy) = u(x,y) + iv(x,y)$ is analytic if u and v satisfy the Cauchy-Riemann equations and have continuous partial derivatives. But since our domain D is $\mathbb{C}\setminus\{0\}$ this is true, and therefore f is analytic in $\mathbb{C}\setminus\{0\}$.

6 (a) The function $f(z) = \frac{z}{(z^2+1)^2}$ have singularities at $z = i, -i$. So we can write

$$f(z) = \frac{z}{(z^2+1)^2} = \frac{z}{((z-i)(z+i)^2} = \frac{z}{(z-i)^2(z+i)^2}\,.$$

Whence $z = i$ and $z = -i$ are singularities of order 2.

Observe that both singularities are inside the circle $|z| = 2$, and hence by the Residue Integration Method it follows that

$$\oint_{|z|=2} f(z)dz = 2\pi i(\mathrm{Res}_{z=i} f(z) + \mathrm{Res}_{z=-i} f(z))\,.$$

So we have to compute the residues of the singularities. Since both singularities have order 2 we can use the formula

$$\mathbf{Res}_{z=z_0} = \lim_{z\to z_0} [(z-z_0)^2 f(z)]'\,.$$

Therefore

$$\mathbf{Res}_{z=i} = \lim_{z\to i}[(z-i)^2 f(z)]' = \lim_{z\to i}\left(\frac{z}{(z+i)^2}\right)' = \lim_{z\to i}\frac{-z+i}{(z+i)^3} = 0\,.$$

and

$$\mathbf{Res}_{z=-i} = \lim_{z\to -i}[(z+i)^2 f(z)]' = \lim_{z\to -i}\left(\frac{z}{(z-i)^2}\right)' = \lim_{z\to -i}\frac{z+i}{(z-i)^3} = 0\,.$$

Thus,

$$\oint_{|z|=2} f(z)dz = 2\pi i(\mathrm{Res}_{z=i} f(z) + \mathrm{Res}_{z=-i} f(z)) = 2\pi i(0+0) = 0\,.$$

(b) The function $f(z) = \frac{z}{(z^2+1)^2}$ has only two singularities $z = i$ and $z = -i$. Observe that the distance between i and $-i$ is 2, i.e., $|i-(-i)| = |2i| = 2$. Then there are two Laurent series of $f(z)$ with center $z = i$: one with convergence region $0 < |z-i| < 2$, and the other with convergence region $|z-i| > 2$. Then since $|2i-i| = |i| = 1 < 2$ we have that for $z = 2i$ the first Laurent series converges, i.e., the one with convergence region $|z-i| < 2$.

7 Let $f(x) = \frac{x^2}{1+x^4}$, we want to compute

$$\int_0^\infty \frac{x^2}{1+x^4} dx \,.$$

Let C_R be the closed path that is the upper-half circle with center the origo and radius R. Observe that we can decompose the path C_R in two parts: the horizontal line $[-R, R]$ and the arc S_R. Then

$$\int_{C_R} \frac{z^2}{1+z^4} dz = \int_{-R}^{R} \frac{x^2}{1+x^4} dx + \int_{S_R} \frac{z^2}{1+z^4} dz \,.$$

Observe that $f(z) = \frac{z^2}{1+z^4}$ has singularities the solutions of $z^4 + 1 = 0$, that are $w_1 = e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$, $w_2 = e^{i\frac{\pi}{4}+\frac{\pi}{2}} = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$, $w_3 = e^{i\frac{\pi}{4}+\pi} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$ and $w_4 = e^{i\frac{\pi}{4}+\frac{3\pi}{2}} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$. All this singularities have order 1.

Using the Residue integration method when $R > 1$ we have that

$$\int_{C_R} \frac{z^2}{1+z^4} dz = 2\pi i (\mathbf{Res}_{z=w_1} f(z) + \mathbf{Res}_{z=w_2} f(z)) \,.$$

Now we compute the residues. Since all the singularities have order 1 we can apply the formula in Page 721 in Kreyzsig to get

$$\mathbf{Res}_{z=w_1} \frac{z^2}{z^4+1} = \frac{z^2}{(z^4+1)'}\Bigg]_{z=w_1} = \frac{z^2}{4z^3}\Bigg]_{z=w_1} = \frac{1}{4z}\Bigg]_{z=w_1} = \frac{1}{4e^{i\frac{\pi}{4}}} = \frac{1}{4}e^{-i\frac{\pi}{4}} = \frac{1}{4}(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})$$

and

$$\mathbf{Res}_{z=w_2} \frac{z^2}{z^4+1} = \frac{z^2}{(z^4+1)'}\Bigg]_{z=w_2} = \frac{z^2}{4z^3}\Bigg]_{z=w_2} = \frac{1}{4z}\Bigg]_{z=w_2} = \frac{1}{4e^{i\frac{3\pi}{4}}} = \frac{1}{4}e^{-i\frac{3\pi}{4}} = \frac{1}{4}(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})$$

Then

$$\int_{C_R} \frac{z^2}{1+z^4} dz = 2\pi i \left(\frac{1}{4}(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}) - \frac{1}{4}(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) \right) = 2\pi i \left(-i\frac{1}{2\sqrt{2}} \right) = \frac{\pi}{\sqrt{2}} \,.$$

Now using the ML-inequality we have that

$$\left| \int_{S_R} \frac{z^2}{1+z^4} dz \right| \leq \mathrm{Max}_{z \in S_R} \left| \frac{z^2}{z^4+1} \right| \cdot \text{Length } S_R \leq \frac{R^2}{R^4} \cdot \pi R = \frac{\pi}{R} \,.$$

Then when have that

$$\lim_{R\to\infty}\left|\int\limits_{S_R}\frac{z^2}{1+z^4}dz\right| = 0\,,$$

and hence

$$\lim_{R\to\infty}\int\limits_{C_R}\frac{z^2}{1+z^4}dz = \lim_{R\to\infty}\left(\int\limits_{-R}^{R}\frac{x^2}{1+x^4}dx + \int\limits_{S_R}\frac{z^2}{1+z^4}dz\right)\,.$$

Thus,

$$\sqrt{2}\pi = \int\limits_{-\infty}^{\infty}\frac{x^2}{1+x^4}dx + 0\,.$$

Finally, since $f(x) = \frac{x^2}{x^4+1}$ is an even function we have that

$$\sqrt{2}\pi = 2\int\limits_{0}^{\infty}\frac{x^2}{1+x^4}dx\,,$$

so

$$\int\limits_{0}^{\infty}\frac{x^2}{1+x^4}dx = \frac{\sqrt{2}}{2}\,.$$