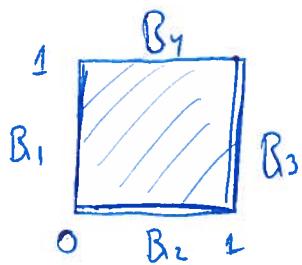


(1)

Example Find the steady-state solution of the square plate



with boundary conditions

$$\underline{B_1} \quad u(0,y) = 0 \quad 0 \leq y \leq 1$$

$$\underline{B_2} \quad u(x,0) = 0 \quad 0 \leq x \leq 1$$

$$\underline{B_3} \quad u_x(1,y) = 0 \quad 0 \leq y \leq 1$$

$$\underline{B_4} \quad u(x,1) = f(x)$$

The Laplace equation says $u_{xx} + u_{yy} = 0$

We want to find solutions of the type

$$u(x,y) = F(x)G(y)$$

Then $u_x = F'_x G \rightarrow u_{xx} = F''_{xx} \cdot G$ $\left\{ \begin{array}{l} \\ (*) \end{array} \right.$

$$u_y = F \cdot G'_y \rightarrow u_{yy} = F \cdot G''_{yy}$$

$$\underline{B_1} \quad u(0,y) = F(0) \cdot G(y) = 0 \rightarrow \boxed{F(0)=0}$$

$$\underline{B_2} \quad u(x,0) = F(x) \cdot G(0) = 0 \Rightarrow \boxed{G(0)=0}$$

$$\underline{B_3} \quad u_x(1,y) = F'_x(1) \cdot G(y) = 0 \Rightarrow \boxed{F'_x(1)=0}$$

Now we replace (*) in the Laplace equation

$$0 = u_{xx} + u_{yy} = F''_{xx} \cdot G + F \cdot G''_{yy}$$

Then $0 = \frac{F''_{xx} \cdot G}{F \cdot G} + \frac{F \cdot G''_{yy}}{F \cdot G} = \frac{F''_{xx}}{F} + \frac{G''_{yy}}{G}$

(2)

Then

$$\frac{F_{xx}}{F} = -\frac{G_{yy}}{G} = k$$

so

$$\begin{cases} F_{xx} - k \cdot F = 0 & (1) \\ G_{yy} + k \cdot G = 0 & (2) \end{cases}$$

We solve (1):

It is easy to check that $\underline{k=0}$ & $\underline{k \geq 0}$ not possibleWe show $k = -p^2 < 0$, then

$$F_{xx} + p^2 \cdot F = 0 \Rightarrow F(x) = A \cdot \cos px + B \sin px$$

so using (B₁) & (B₂)

$$0 = F(0) = A \quad \& \quad 0 = F_x(0) = B \cdot p \cos p = 0$$

$$\text{Then } p = \frac{\pi}{2} + n\pi \quad n=0, 1, 2, \dots$$

$$p = \frac{\pi + 2n\pi}{2} = \frac{(2n+1)\pi}{2}$$

$$\text{Then } \boxed{F_n(x) = A_n \cdot \sin \left(\frac{(2n+1)\pi}{2} x \right)} \quad n=0, 1, 2, 3, \dots$$

Now solve (2):

$$G_{yy} - p^2 \cdot G = 0 \Rightarrow G(y) = B e^{py} + C e^{-py}$$

so using (B₂)

$$0 = G(0) = B + C \Rightarrow B = -C$$

$$\text{Then } \boxed{G_n(y) = B_n \left(e^{\left(\frac{(2n+1)\pi}{2}\right) py} - e^{-\left(\frac{(2n+1)\pi}{2}\right) py} \right) = B_n \cdot \sinh \left(\left(\frac{2n+1}{2}\right) \pi y \right)}$$

(3)

Therefore

$$u_n(x, y) = C_n \cdot \sin\left(\frac{(2n+1)\pi}{2}x\right) \cdot \sinh\left(\frac{(2n+1)\pi}{2}y\right)$$

Now we will find C_n 's such that

$$u(x, 1) = \sum_{n=0}^{\infty} u_n(x, 1) = f(x) \quad \text{for } 0 \leq x \leq L$$

But

$$u_n(x, 1) = C_n \cdot \sin\left(\frac{(2n+1)\pi}{2}x\right) \cdot \sinh\left(\frac{(2n+1)\pi}{2}\right)$$

so

$$\sum_{n=0}^{\infty} C_n \cdot \underbrace{\sin\left(\frac{(2n+1)\pi}{2}x\right)}_{\text{have period } 4} \cdot \sinh\left(\frac{(2n+1)\pi}{2}\right)$$

$\Rightarrow L=2$

observe

$$\sin\frac{\pi}{2}x, \cancel{\sin\pi x}, \sin\frac{3\pi}{2}x, \cancel{\sin 2\pi x}, \sin\frac{5\pi}{2}x, \dots$$

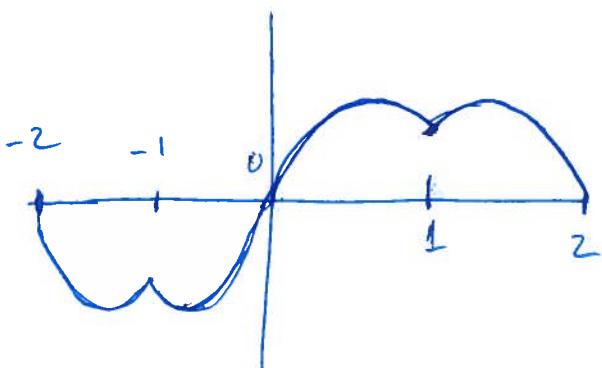
so we want to find D_n such that

$$f(x) = \sum_{n=1}^{\infty} D_n \cdot \sin\frac{n\pi}{2}x \quad \text{for } 0 \leq x \leq L.$$

$$\text{with } D_2 = D_4 = D_6 = \dots = 0$$

(4)

Define the function $g(x)$ 4-periodic



$$g(x) = \begin{cases} f(-x+2) & 1 \leq x \leq 2 \\ f(x) & 0 \leq x \leq 1 \\ -f(-x) & -1 \leq x \leq 0 \\ -f(x+2) & -2 \leq x \leq -1 \end{cases}$$

$g(x)$ is an odd-function, so

$$g(x) = \sum_{n=1}^{\infty} D_n \cdot \sin \frac{n\pi}{2} x$$

~~Since $f(x)$ is even~~

so $D_n = \frac{2}{2} \cdot \int_0^2 g(x) \sin \frac{n\pi}{2} x \, dx$

It is easy to check that $D_2 = D_4 = D_6 = \dots = 0$.

Moreover, $g(x) = f(x)$ when $0 \leq x \leq 1$.

Then

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} D_n \cdot \sin \frac{(2n+1)\pi}{2} x \\ &= \sum_{n=0}^{\infty} C_n \cdot \sin \frac{(2n+1)\pi}{2} x \cdot \sinh \left(\frac{(2n+1)\pi}{2} \right) \end{aligned}$$

so

$$D_n = C_n \cdot \sinh \left(\frac{(2n+1)\pi}{2} \right) \Rightarrow$$

$$C_n = \frac{D_n}{\sinh \left(\frac{(2n+1)\pi}{2} \right)}$$