LF MATTE 4K KONT 2015

1) LØS INITIALVERDIPROBLEMET

$$y''(t) - 4y'(t) + 3y(t) = \delta(t-5), \quad y(0) = y'(0) = 1$$

·) $\mathcal{L}(y(t))(s) := Y(s)$

·) $\mathcal{L}(y'(t))(s) = s\mathcal{L}(y(t))(s) - y(0) = sY(s) - 1$

·) $\mathcal{L}(y''(t))(s) = s\mathcal{L}(y'(t))(s) - y'(0) = s(s\mathcal{L}(y(t))(s) - y(0)) - y'(0) = s^2Y(s) - s - 1$

·) $\mathcal{L}(\delta(t-5))(s) = e^{-5s}$

$\rightsquigarrow$ $y''(t) - 4y'(t) + 3y(t) = \delta(t-5), \quad y(0) = y'(0) = 1$

$\Downarrow$ LAPLACETRANSF

$s^2Y(s) - s - 1 - 4sY(s) + 4 + 3Y(s) = e^{-5s}$

$(s^2 - 4s + 3)Y(s) = s - 3 + e^{-5s}$

$$Y(s) = \frac{s-3+e^{-5s}}{(s-3)(s-1)}$$

$\rightsquigarrow Y(s) = \frac{1}{s-1} + \frac{e^{-5s}}{(s-3)(s-1)}$

NULLPUNKTER TIL s^2-4s+3

$s_{1,2} = 2 \pm \sqrt{4-3} = 2 \pm 1$

$\rightsquigarrow (s^2-4s+3) = (s-3)(s-1)$

DELBRØKOPPSP

$$\frac{1}{(s-3)(s-1)} = \frac{A}{s-3} + \frac{B}{s-1}$$

$\rightsquigarrow 1 = (s-1)A + (s-3)B = s(A+B) - (A+3B)$

$\rightarrow A + B = 0 \rightarrow A = -B$

$A + 3B = -1$

$2B = -1 \Rightarrow B = -\frac{1}{2} \quad A = \frac{1}{2}$

$$Y(s) = \frac{1}{s-1} + \frac{1}{2}\left(\frac{1}{s-3} - \frac{1}{s-1}\right)e^{-5s}$$

·) $\mathcal{L}^{-1}\left(\frac{1}{s-a}\right)(t) = e^{at}$

$\mathcal{L}^{-1}(e^{-as}F(s))(t) = f(t-a)u(t-a) \qquad [f(t) = \mathcal{L}^{-1}(F(s))(t)]$

$\rightsquigarrow$ ·) $\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) = e^{3t} \rightsquigarrow \mathcal{L}^{-1}\left(e^{-5s}\frac{1}{s-3}\right) = e^{3(t-5)}u(t-5)$

·) $\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^{t} \rightsquigarrow \mathcal{L}^{-1}\left(e^{-5s}\frac{1}{s-1}\right) = e^{t-5}u(t-5)$

$$Y(s) = \frac{1}{s-1} + \frac{1}{2}\left(\frac{1}{s-3} - \frac{1}{s-1}\right)e^{-5s}$$

$\Downarrow$ LAPLACETRANSF^{-1}

$$y(t) = e^t + \frac{1}{2}\left(e^{3(t-5)} - e^{t-5}\right)u(t-5)$$

ANNEN MULIGHET: $Y(s) = \frac{1}{s-1} + \frac{e^{-5s}}{(s-1)(s-3)} = \frac{1}{s-1} + \frac{e^{-5s}}{s^2-4s+3} = \frac{1}{s-1} + \frac{e^{-5s}}{(s-2)^2-1}$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right)(t) = \sinh(t)$$

$$\mathcal{L}^{-1}(F(s-a))(t) = e^{at} f(t) \quad \rightsquigarrow \quad \mathcal{L}^{-1}\left(\frac{1}{(s-2)^2-1}\right)(t) = e^{2t}\sinh(t)$$

$$\mathcal{L}^{-1}\left(\frac{e^{-5s}}{(s-2)^2-1}\right)(t) = u(t-5)\, e^{2(t-5)} \sinh(t-5)$$

2) LA FUNKSJONEN f VÆRE DEFINERT VED $f(x)=\cos(x)$ FOR $0<x<\pi$

a) FINN FOURIERSINUSREKKEN TIL $f(x)$

$\cos'(x) = -\sin(x)$
$\sin'(x) = \cos(x)$

$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$ MED $b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx$

$$\Rightarrow b_n = \frac{2}{\pi}\int_0^\pi \sin(nx)\cos(x)\,dx = \frac{2}{\pi}\sin(nx)\sin(x)\Big|_0^\pi - \frac{2}{\pi}n\int_0^\pi \cos(nx)\sin(x)\,dx$$

$$= \frac{2}{\pi}n\cos(nx)\cos(x)\Big|_0^\pi + \frac{2}{\pi}n^2\int_0^\pi \sin(nx)\cos(x)\,dx$$

$$= \frac{2}{\pi}n(-1)^{n+1} - \frac{2}{\pi}n + n^2 b_n.$$

$\Rightarrow n \neq 1$: $\quad b_n = \frac{2}{\pi}\frac{n}{1-n^2}\left((-1)^{n+1}-1\right) = \frac{2}{\pi}\frac{n}{n^2-1}\left((-1)^n+1\right)$

b_1: $\quad b_1 = \frac{2}{\pi}\int_0^\pi \sin(x)\cos(x)\,dx = \frac{2}{\pi}\sin(x)^2\Big|_0^\pi - \frac{2}{\pi}1\int_0^\pi \cos(x)\sin(x)\,dx = -b_1$

$\Rightarrow b_1 = 0$

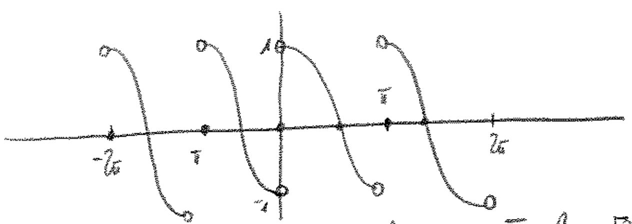
$$\Rightarrow f(x) = \sum_{n=2}^{\infty} \frac{2}{\pi}\frac{n}{n^2-1}\left((-1)^n+1\right)\sin(nx)$$

$$= \sum_{\ell=1}^{\infty} \frac{2}{\pi}\frac{2\ell}{(2\ell)^2-1}\cdot 2\sin(2\ell x) = \frac{8}{\pi}\sum_{\ell=1}^{\infty}\frac{\ell}{(2\ell)^2-1}\sin(2\ell x)$$

b) SKISSER SUMMEN AV FOURIERSINUSREKKEN TIL $f(x)$ PÅ INTERVALLET $[-2\pi, 2\pi]$
FINN VERDIEN TIL FOURIERSINUSREKKEN TIL $f(x)$ I PUNKTENE $x = -\frac{\pi}{4}$, $x=0$ OG $x = \frac{\pi}{2}$

FOURIERSINUSREKKEN TIL $f(x) \approx$ ODDE UTVIDELSE TIL $f(x)$.

$\Rightarrow$ FOURIERSINUSREKKEN TIL $f(x)$



LA $F(x)$ ... FOURIERSINUSREKKEN TIL $f(x)$ $\rightsquigarrow$ $F(x) = f(x)$ FOR $x = (0, \pi)$. (f KONT PÅ $(0,\pi)$)

$\Rightarrow F\left(-\frac{\pi}{4}\right) = -F\left(\frac{\pi}{4}\right) = -f\left(\frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{2}$

$F\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0.$

$F(0) = \frac{1}{2}(F(0+) + F(0-)) = \frac{1}{2}(F(0+) - F(0+)) = 0.$ (F DISKONT I $x=0$)

3) LA C VÆRE SIRKELEN $\{z\in\mathbb{C} : |z-2|=2\}$. ORIENTERT MOT KLOKKA. FINN VERDIEN AV LINJEINTEGRALET

$$\oint_C \frac{1}{(z-1)(z-7)}\,dz$$

INTEGRAND $\frac{1}{(z-1)(z-7)}$: TO POLER AV ORDNING 1. $z=1, z=7$.

$|1-2|=|1|<2 \Rightarrow z=1$ PÅ INNSIDEN

$|7-2|=5>2 \Rightarrow z=7$ PÅ UTSIDEN

$$\Rightarrow \oint_C \frac{1}{(z-1)(z-7)}\,dz = 2\pi i \operatorname*{Res}_{z=1} \frac{1}{(z-1)(z-7)} = 2\pi i \frac{1}{1-7} = -\frac{\pi i}{3}$$

4) VIS VED HJELP AV CAUCHY-RIEMANNLIGNINGENE AT $f(z)=ze^{iz}$ ER EN HEL FUNKSJON, DVS AT $f(z)$ ER ANALYTISK I HELE $\mathbb{C}$

BRUK THM 13.4.2.

$$f(z)=ze^{iz}=(x+iy)e^{i(x+iy)}=(x+iy)e^{-y}e^{ix}=e^{-y}(x+iy)(\cos(x)+i\sin(x))$$
$$=(x\cos(x)-y\sin(x))e^{-y}+i(y\cos(x)+x\sin(x))e^{-y}$$

$\Rightarrow u(x,y)=(x\cos(x)-y\sin(x))e^{-y} \rightsquigarrow u_x(x,y)=(\cos(x)-x\sin(x)-y\cos(x))e^{-y}$ $\rightsquigarrow$ KONT

$u_y(x,y)=(-\sin(x)-x\cos(x)+y\sin(x))e^{-y}$ $\rightsquigarrow$ KONT

$v(x,y)=(y\cos(x)+x\sin(x))e^{-y} \rightsquigarrow v_x(x,y)=(-y\sin(x)+\sin(x)+x\cos(x))e^{-y}$ $\rightsquigarrow$ KONT

$v_y(x,y)=(\cos(x)-y\cos(x)-x\sin(x))e^{-y}$ $\rightsquigarrow$ KONT

$u_x(x,y)=(\cos(x)-x\sin(x)-y\cos(x))e^{-y}=v_y(x,y)$ ✓

$u_y(x,y)=(-\sin(x)-x\cos(x)+y\sin(x))e^{-y}=-v_x(x,y)$ ✓

5) LA $R>0$ OG S_R VÆRE HALVSIRKELEN MED PARAMETRISERING $z(\theta)=Re^{i\theta}$ $0\le\theta\le\pi$.

LA $x\ge 0$ OG BRUK ML-ULIKHETEN TIL Å VISE AT

$$\lim_{R\to\infty}\int_{S_R}\frac{1}{4+w^4}e^{iwx}\,dw=0.$$

$$\int_{S_R}\frac{1}{4+w^4}e^{+iwx}\,dw=\int_0^{\pi}\frac{1}{4+R^4e^{4i\theta}}e^{+iR(\cos(\theta)+i\sin\theta)}\,iRe^{i\theta}\,d\theta$$

$$\Rightarrow \left|\int_{S_R}\frac{1}{4+w^4}e^{+iwx}\,dw\right|\le\pi\frac{1}{(R^4-4)}R\to 0 \quad (R\to\infty)$$

BRUKTE $|e^{i\theta}|=1$ OG $-R\sin\theta<0$. OG DERMED $|e^{iR(\cos(\theta)+i\sin\theta)}|\le e^{-R\sin(\theta)}\le 1$.

6) a) FINN OG KLASSIFISER DE SINGULÆRE PUNKTENE TIL FUNKSJONEN

$$f(z) = \frac{z^{4n}-1}{z^{2n+1}} \quad n = 1, 2, \dots$$

FINN LAURENTREKKEN TIL $f(z)$ OM $z=0$ OG REGN UT RESIDYET I $z=0$.

$$f(z) = \frac{z^{4n}-1}{z^{2n+1}} = z^{2n-1} - \frac{1}{z^{2n+1}} \Rightarrow$$

SINGULÆRE PUNKTENE
$z=0$: POL AV ORDEN $2n+1$.

LAURENTREKKEN TIL $f(z)$: $-\frac{1}{z^{2n+1}} + z^{2n-1}$

b) BRUK RESIDYREGNING FOR Å VISE AT

$$\int_0^{2\pi} \cos(n\theta)\sin(n\theta)\,d\theta = 0 \quad n = 1, 2, \dots$$

BRUK SUBSTITUSJON $z = e^{i\theta} = \cos(\theta) + i\sin(\theta)$
$\Rightarrow e^{-i\theta} = \cos(\theta) - i\sin(\theta)$

$$\Rightarrow \cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\Rightarrow \cos(n\theta) = \frac{1}{2}\left(e^{in\theta} + e^{-in\theta}\right) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$$

$$\sin(n\theta) = \frac{1}{2i}\left(e^{in\theta} - e^{-in\theta}\right) = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)$$

$$\frac{dz}{d\theta} = ie^{i\theta} = iz$$

$$\Rightarrow \int_0^{2\pi} \cos(n\theta)\sin(n\theta)\,d\theta = \oint_C \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)\frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)\frac{1}{iz}\,dz = \oint_C -\frac{1}{4}\underbrace{\left(z^{2n-1} - \frac{1}{z^{2n+1}}\right)}_{f(z) \text{ FRA A}} dz$$

$C \downarrow$ ENHETSSIRKEL MOT KLOKKA

$$= -2\pi i \frac{1}{4} b_1$$

MED b_1 = KOEFF TIL $\frac{1}{z}$ I LAURENTREKKEN TIL $f(z)$.

$2n-1 \geq 1$ FOR $n = 1, 2, \dots$

$2n+1 \geq 3$ FOR $n = 1, 2, \dots$

$$\Downarrow$$

$$b_1 = 0$$

$$\Rightarrow \int_0^{2\pi} \cos(n\theta)\sin(n\theta)\,d\theta = 0$$

TIL 6a) $\operatorname{Res} f(z)$ = KOEFFISIENTEN TIL $\frac{1}{z}$ I LAURENTREKKEN
$= 0$

7a) FINN ALLE LØSNINGER PÅ FORMEN $u(x,t) = F(x)G(t)$ SOM TILFREDSSTILLER DEN PARTIELLE DIFFERENSIALLIGNINGEN

$$u_t(x,t) + (1+t^2)(2u(x,t) - u_{xx}(x,t)) = 0 \quad x \in [0, \tfrac{\pi}{2}],\ t>0$$

OG RANDBETINGELSENE:

$$u(0,t) = u_x(\tfrac{\pi}{2},t) = 0 \quad t \geq 0.$$

$$u(x,t) = F(x)G(t) \rightarrow u(0,t) = F(0)G(t) = 0 \ \forall t \geq 0 \Rightarrow \begin{cases} F(0)=0 \ \checkmark \\ G(t) = 0 \ \forall t \geq 0 \rightsquigarrow u \equiv 0 \ (\text{UØNSKET}) \end{cases}$$

$$u_x(\tfrac{\pi}{2},t) = F'(\tfrac{\pi}{2})G(t) \ \forall t \geq 0 \Rightarrow \begin{cases} F'(\tfrac{\pi}{2}) = 0 \ \checkmark \\ G(t) = 0 \ \forall t \geq 0 \rightsquigarrow u \equiv 0 \ (\text{UØNSKET}). \end{cases}$$

$\rightsquigarrow$ RANDBETINGELSENE : $F(0) = 0 = F'(\tfrac{\pi}{2})$

$$u_t(x,t) + (1+t^2)(2u(x,t) - u_{xx}(x,t)) = 0 \rightsquigarrow F(x)G'(t) + (1+t^2)(2F(x)G(t) - F''(x)G(t)) = 0$$

$$\frac{G'(t)}{(1+t^2)G(t)} = \frac{F''(x) - 2F(x)}{F(x)} = \frac{F''(x)}{F(x)} - 2 = \lambda \quad \text{KONSTANT}$$

•) $\dfrac{F''(x)}{F(x)} = \lambda + 2 \Rightarrow (\lambda+2)F(x) = F''(x)$

•) $\lambda + 2 = 0 \Rightarrow F''(x) = 0 \Rightarrow F(x) = Ax + B$

$$\left.\begin{aligned} F(0) = 0 &\Rightarrow B = 0 \\ F'(\tfrac{\pi}{2}) = 0 &\Rightarrow A = 0 \end{aligned}\right\} \Rightarrow u \equiv 0 \ (\text{UØNSKET})$$

•) $(\lambda+2) > 0 \Rightarrow F(x) = Ae^{\sqrt{2+\lambda}\,x} + Be^{-\sqrt{2+\lambda}\,x}$

$$\left.\begin{aligned} F(0) = 0 &\Rightarrow A + B = 0 \\ F'(\tfrac{\pi}{2}) = 0 &\Rightarrow A\left(e^{\sqrt{2+\lambda}\frac{\pi}{2}} + e^{-\sqrt{2+\lambda}\frac{\pi}{2}}\right) = 0 \end{aligned}\right\} \Rightarrow F \equiv 0 \Rightarrow u \equiv 0 \ (\text{UØNSKET})$$

•) $(\lambda+2) < 0 \Rightarrow F(x) = A\cos(\sqrt{-(\lambda+2)}\,x) + B\sin(\sqrt{-(\lambda+2)}\,x)$

$F(0) = 0 \Rightarrow A = 0$

$$F'(\tfrac{\pi}{2}) \Rightarrow B\cos(\sqrt{-(\lambda+2)}\,\tfrac{\pi}{2}) = 0 \Rightarrow \begin{cases} B = 0 \Rightarrow u \equiv 0 \ (\text{UØNSKET}) \\ B \neq 0 \text{ OG } \sqrt{-(\lambda+2)} = (2n+1) \quad n = 0, 1, 2, \dots \end{cases}$$

$$\Rightarrow \lambda + 2 = -(2n+1)^2$$

$$\lambda = -(2n+1)^2 - 2$$

$\Rightarrow$ LET $F_n(x) = B\sin((2n+1)x)$

$\Rightarrow G_n(t)$ OPPFYLLER: $\dfrac{G_n'(t)}{(1+t^2)G_n(t)} = \lambda = -(2n+1)^2 - 2$

$$\rightsquigarrow G_n(t) = G_n(0)e^{\lambda(t + \frac{t^3}{3})}$$

LØSNINGER PÅ FORMEN

$$u(x,t) = \sum_{n=0}^{\infty} B_n \sin((2n+1)x)e^{-((2n+1)^2+2)(t + \frac{t^3}{3})}.$$

b) FINN EN LØSNING SOM TILFREDSSTILLER (1) OG (2) OG I TILLEGG RANDBETINGELSEN:

$$u(x,0) = \sin(3x) + \sin(17x) \qquad x \in [0, \tfrac{\pi}{2}]$$

$$u(x,0) = \sum_{n=0}^{\infty} B_n \sin((2n+1)x) \quad \Rightarrow \quad \begin{array}{l} B_1 = 1 \\ B_8 = 1 \\ \text{ELLERS } B_i = 0. \end{array}$$

$$\Rightarrow u(x,t) = \sin(3x) e^{-11(t+\frac{t^3}{3})} + \sin(17x) e^{-291(t+\frac{t^3}{3})}$$