LF MATTE 4K KONT 2015

1) LØS INITIALVERDIPROBLEMET

$$y''(t) - 4y'(t) + 3y(t) = \delta(t-5), \quad y(0) = y'(0) = 1$$

$$\cdot\ \mathcal{L}(y(t))(s) = Y(s)$$

$$\cdot\ \mathcal{L}(y'(t))(s) = s\mathcal{L}(y(t))(s) - y(0) = sY(s) - 1$$

$$\cdot\ \mathcal{L}(y''(t))(s) = s\mathcal{L}(y'(t))(s) - y'(0) = s\left(s\mathcal{L}(y(t))(s) - y(0)\right) - y'(0) = s^2Y(s) - s - 1$$

$$\cdot\ \mathcal{L}(\delta(t-5))(s) = e^{-5s}$$

$$\Rightarrow \quad y''(t) - 4y'(t) + 3y(t) = \delta(t-5), \quad y(0) = y'(0) = 1$$

$$\Big\downarrow \text{LAPLACETRANSF}$$

$$s^2Y(s) - s - 1 - 4sY(s) + 4 + 3Y(s) = e^{-5s}$$

$$(s^2 - 4s + 3)Y(s) = s - 3 + e^{-5s}$$

$$Y(s) = \frac{s-3+e^{-5s}}{(s-3)(s-1)}$$

NULLPUNKTER TIL $s^2 - 4s + 3$

$$s_{1,2} = 2 \pm \sqrt{4-3} = 2 \pm 1$$

$$\Rightarrow (s^2 - 4s + 3) = (s-3)(s-1)$$

$$\Rightarrow Y(s) = \frac{1}{s-1} + \frac{e^{-5s}}{(s-3)(s-1)}$$

DELBRØKOPPSP.

$$\frac{1}{(s-3)(s-1)} = \frac{A}{s-3} + \frac{B}{s-1}$$

$$\Rightarrow 1 = (s-1)A + (s-3)B = s(A+B) - (A+3B)$$

$$\Rightarrow A + B = 0 \Rightarrow A = -B$$
$$A + 3B = -1 \quad \Downarrow$$
$$2B = -1 \Rightarrow B = -\tfrac{1}{2}, \quad A = \tfrac{1}{2}$$

$$Y(s) = \frac{1}{s-1} + \frac{1}{2}\left(\frac{1}{s-3} - \frac{1}{s-1}\right)e^{-5s}$$

$$\cdot\ \mathcal{L}^{-1}\left(\frac{1}{s-a}\right)(t) = e^{at}$$

$$\mathcal{L}^{-1}\left(e^{-as}F(s)\right)(t) = f(t-a)u(t-a) \qquad \left[\, f(t) = \mathcal{L}^{-1}(F(s))(t) \,\right]$$

$$\Rightarrow \cdot\ \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) = e^{3t} \Rightarrow \mathcal{L}^{-1}\left(e^{-5s}\frac{1}{s-3}\right) = e^{3(t-5)}u(t-5)$$

$$\cdot\ \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^{t} \Rightarrow \mathcal{L}^{-1}\left(e^{-5s}\frac{1}{s-1}\right) = e^{t-5}u(t-5)$$

$$Y(s) = \frac{1}{s-1} + \frac{1}{2}\left(\frac{1}{s-3} - \frac{1}{s-1}\right)e^{-5s}$$

$$\Big\downarrow \text{LAPLACETRANSF}^{-1}$$

$$y(t) = e^{t} + \frac{1}{2}\left(e^{3(t-5)} - e^{t-5}\right)u(t-5)$$

ANNEN MULIGHET

$$Y(s)=\frac{1}{s-1}+\frac{e^{-5s}}{(s-1)(s-3)}=\frac{1}{s-1}+\frac{e^{-5s}}{s^2-4s+3}=\frac{1}{s-1}+\frac{e^{-5s}}{(s-2)^2-1}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right)(t)=\sinh(t)$$

$$\mathcal{L}^{-1}(F(s-a))(t)=e^{at}f(t) \Rightarrow \mathcal{L}^{-1}\left(\frac{1}{(s-2)^2-1}\right)(t)=e^{2t}\sinh(t)$$

$$\mathcal{L}^{-1}\left(\frac{e^{-5s}}{(s-2)^2-1}\right)(t)=u(t-5)\,e^{2(t-5)}\sinh(t-5)$$

2) LA FUNKSJONEN f VÆRE DEFINERT VED $f(x)=\cos(x)$ FOR $0<x<\pi$

a) FINN FOURIERSINUSREKKEN TIL $f(x)$

$$f(x)=\sum_{n=1}^{\infty} b_n\sin(nx) \quad \text{MED} \quad b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$\cos'(x)=-\sin(x)$
$\sin'(x)=\cos(x)$

$$\Rightarrow b_n=\frac{2}{\pi}\int_0^{\pi}\sin(nx)\cos(x)\,dx=\frac{2}{\pi}\sin(nx)\sin(x)\Big|_0^{\pi}-\frac{2}{\pi}n\int_0^{\pi}\cos(nx)\sin(x)\,dx$$

$$=\frac{2}{\pi}n\cos(nx)\cos(x)\Big|_0^{\pi}+\frac{2}{\pi}n^2\int_0^{\pi}\sin(nx)\cos(x)\,dx$$

$$=\frac{2}{\pi}n(-1)^{n+1}-\frac{2}{\pi}n+n^2b_n$$

$$\Rightarrow n\neq 1 \qquad b_n=\frac{2}{\pi}\frac{n}{1-n^2}\left((-1)^{n+1}-1\right)=\frac{2}{\pi}\frac{n}{n^2-1}\left((-1)^n+1\right)$$

$$b_1=\frac{2}{\pi}\int_0^{\pi}\sin(x)\cos(x)\,dx=\frac{2}{\pi}\sin(x)^2\Big|_0^{\pi}-\frac{2}{\pi}\int_0^{\pi}\cos(x)\sin(x)\,dx=-b_1$$

$$\Rightarrow b_1=0$$

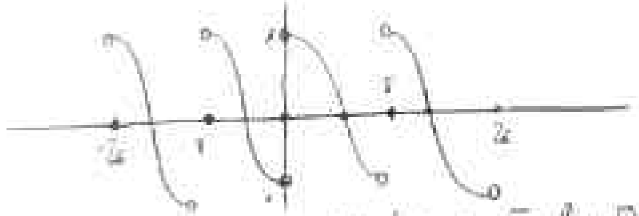
$$\Rightarrow f(x)=\sum_{n=2}^{\infty}\frac{2}{\pi}\frac{n}{n^2-1}\left((-1)^n+1\right)\sin(nx)$$

$$=\sum_{\ell=1}^{\infty}\frac{2}{\pi}\frac{2\ell}{(2\ell)^2-1}\cdot 2\sin(2\ell x)=\frac{8}{\pi}\sum_{\ell=1}^{\infty}\frac{\ell}{(2\ell)^2-1}\sin(2\ell x)$$

b) SKISSER SUMMEN AV FOURIERSINUSREKKEN TIL $f(x)$ PÅ INTERVALLET $[-2\pi,2\pi]$
FINN VERDIEN TIL FOURIERSINUSREKKEN TIL $f(x)$ I PUNKTENE $x=-\frac{\pi}{4}$, $x=0$ OG $x=\frac{\pi}{2}$

FOURIERSINUSREKKEN TIL $f(x)$ ≈ ODDE UTVIDELSE TIL $f(x)$

⇒ FOURIERSINUSREKKEN TIL $f(x)$



LA $F(x)$ FOURIERSINUSREKKEN TIL $f(x)$ ⇒ $F(x)=f(x)$ FOR $x\in(0,\pi)$ (f KONT PÅ $(0,\pi)$)

$$\Rightarrow F\left(-\frac{\pi}{4}\right)=-F\left(\frac{\pi}{4}\right)=-f\left(\frac{\pi}{4}\right)=-\cos\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$$

$$F\left(\frac{\pi}{2}\right)=f\left(\frac{\pi}{2}\right)=\cos\left(\frac{\pi}{2}\right)=0$$

$$F(0)=\frac{1}{2}\left(F(0+)+F(0-)\right)=\frac{1}{2}\left(F(0+)-F(0+)\right)=0 \quad (F \text{ DISKONT I } x=0)$$

3) LA C VÆRE SIRKELEN $\{z \in \mathbb{C} : |z-2|=2\}$ ORIENTERT MOT KLOKKA. FINN VERDIEN AV LINJEINTEGRALET

$$\oint_C \frac{1}{(z-1)(z-7)}\,dz$$

INTEGRAND $\frac{1}{(z-1)(z-7)}$ TO POLER AV ORDNING 1 $z=1, z=7$

$|1-2|=|1|<2 \Rightarrow z=1$ PÅ INNSIDEN

$|7-2|=5>2 \Rightarrow z=7$ PÅ UTSIDEN

$$\Rightarrow \oint_C \frac{1}{(z-1)(z-7)}\,dz = 2\pi i \operatorname*{Res}_{z=1} \frac{1}{(z-1)(z-7)} = 2\pi i \frac{1}{1-7} = -\frac{\pi i}{3}$$

4) VIS VED HJELP AV CAUCHY-RIEMANNLIGNINGENE AT $f(z)=ze^{iz}$ ER EN HEL FUNKSJON, DVS AT $f(z)$ ER ANALYTISK I HELE $\mathbb{C}$

BRUK THM 13.4.2

$$f(z)=ze^{iz}=(x+iy)e^{i(x+iy)}=(x+iy)e^{-y}e^{ix}=e^{-y}(x+iy)(\cos(x)+i\sin(x))$$
$$=(x\cos(x)-y\sin(x))e^{-y}+i(y\cos(x)+x\sin(x))e^{-y}$$

$\rightarrow u(x,y)=(x\cos(x)-y\sin(x))e^{-y} \Rightarrow u_x(x,y)=(\cos(x)-x\sin(x)-y\cos(x))e^{-y}$ → KONT

$u_y(x,y)=(-\sin(x)-x\cos(x)+y\sin(x))e^{-y}$ → KONT

$v(x,y)=(y\cos(x)+x\sin(x))e^{-y} \rightarrow v_x(x,y)=(-y\sin(x)+\sin(x)+x\cos(x))e^{-y}$ → KONT

$v_y(x,y)=(\cos(x)-y\cos(x)-x\sin(x))e^{-y}$ → KONT

$u_x(x,y)=(\cos(x)-x\sin(x)-y\cos(x))e^{-y}=v_y(x,y)$ ✓

$u_y(x,y)=(-\sin(x)-x\cos(x)+y\sin(x))e^{-y}=-v_x(x,y)$ ✓

5) LA $R>0$ OG S_R VÆRE HALVSIRKELEN MED PARAMETRISERING $z(\theta)=Re^{i\theta}$ $0\le\theta\le\pi$

LA $\alpha>0$ OG BRUK ML-ULIKHETEN TIL Å VISE AT

$$\lim_{R\to\infty}\int_{S_R}\frac{1}{4+\omega^4}e^{i\alpha\omega}\,d\omega=0$$

$$\int_{S_R}\frac{1}{4+\omega^4}e^{i\alpha\omega}\,d\omega=\int_0^\pi\frac{1}{4+(Re^{i\theta})^4}e^{i\alpha R(\cos(\theta)+i\sin(\theta))}\,iRe^{i\theta}\,d\theta$$

$$\Rightarrow \left|\int_{S_R}\frac{1}{4+\omega^4}e^{i\alpha\omega}\,d\omega\right| \le \pi\frac{1}{(R^4-4)}R \rightarrow 0 \quad (R\to\infty)$$

BRUKTE $|e^{i\theta}|=1$ OG $-R\sin\theta\le 0$ OG DERMED $|e^{iR(\cos(\theta)+i\sin(\theta))}|\le e^{-R\sin(\theta)}\le 1$

6) a) FINN OG KLASSIFISER DE SINGULÆRE PUNKTENE TIL FUNKSJONEN

$$f(z) = \frac{z^{4n} - 1}{z^{2n+1}} \quad n = 1, 2, \dots$$

FINN LAURENTREKKEN TIL $f(z)$ OM $z = 0$ OG REGN UT RESIDYET I $z = 0$

$$f(z) = \frac{z^{4n} - 1}{z^{2n+1}} = z^{2n-1} - \frac{1}{z^{2n+1}} \rightarrow$$ SINGULÆRE PUNKTENE $z = 0$ POL AV ORDEN $2n+1$

LAURENTREKKEN TIL $f(z)$: $-\frac{1}{z^{2n+1}} + z^{2n-1}$

b) BRUK RESIDYREGNING FOR Å VISE AT

$$\int_0^{2\pi} \cos(n\theta)\sin(n\theta)\,d\theta = 0 \quad n = 1, 2, \dots$$

BRUK SUBSTITUSJON $z = e^{i\theta} = \cos(\theta) + i\sin(\theta)$
$\Rightarrow e^{-i\theta} = \cos(\theta) - i\sin(\theta)$

$$\Rightarrow \cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$
$$\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\Rightarrow \cos(n\theta) = \frac{1}{2}(e^{in\theta} + e^{-in\theta}) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$$
$$\sin(n\theta) = \frac{1}{2i}(e^{in\theta} - e^{-in\theta}) = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)$$

$$\frac{dz}{d\theta} = ie^{i\theta} = iz$$

$$\Rightarrow \int_0^{2\pi} \cos(n\theta)\sin(n\theta)\,d\theta = \oint_C \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)\frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)\frac{1}{iz}\,dz = \oint_C -\frac{1}{4}\underbrace{\left(z^{2n-1} - \frac{1}{z^{2n+1}}\right)}_{f(z) \text{ FRA A}}dz$$

$C \downarrow$ ENHETSSIRKEL MOT KLOKKA

$$= -2\pi i \frac{1}{4} b_1$$

MED b_1 = KOEFF TIL $\frac{1}{z}$ I LAURENTREKKEN TIL $f(z)$

$2n-1 \geq 1$ FOR $n = 1, 2, \dots$

$2n+1 \geq 3$ FOR $n = 1, 2, \dots$

$\Downarrow$

$b_1 = 0$

$$\Rightarrow \int_0^{2\pi} \cos(n\theta)\sin(n\theta)\,d\theta = 0$$

TIL 6a): $\mathrm{Res}_{z=0} f(z)$ = KOEFFISIENTEN TIL $\frac{1}{z}$ I LAURENTREKKEN $= 0$

7a) FINN ALLE LØSNINGER PÅ FORMEN $u(x,t)=F(x)G(t)$ SOM TILFREDSSTILLER DEN PARTIELLE DIFFERENSIALLIGNINGEN

$$u_t(x,t)+(1+t^2)(2u(x,t)-u_{xx}(x,t))=0 \quad x\in(0,\tfrac{\pi}{2}),\ t>0$$

OG RANDBETINGELSENE

$$u(0,t)=u_x(\tfrac{\pi}{2},t)=0 \quad t\geq 0$$

$$u(x,t)=F(x)G(t) \Rightarrow u(0,t)=F(0)G(t)=0 \ \forall t\geq 0 \Rightarrow \begin{cases} F(0)=0 \ ✓ \\ G(t)=0 \ \forall t\geq 0 \Rightarrow u\equiv 0 \ \text{(UØNSKET)} \end{cases}$$

$$u_x(\tfrac{\pi}{2},t)=F'(\tfrac{\pi}{2})G(t) \ \forall t\geq 0 \Rightarrow \begin{cases} F'(\tfrac{\pi}{2})=0 \ ✓ \\ G(t)=0 \ \forall t\geq 0 \Rightarrow u\equiv 0 \ \text{(UØNSKET)} \end{cases}$$

$\Rightarrow$ RANDBETINGELSENE $\quad F(0)=0=F'(\tfrac{\pi}{2})$

$$u_t(x,t)+(1+t^2)(2u(x,t)-u_{xx}(x,t))=0 \Rightarrow F(x)G'(t)+(1+t^2)(2F(x)G(t)-F''(x)G(t))=0$$

$$\frac{G'(t)}{(1+t^2)G(t)}=\frac{F''(x)-2F(x)}{F(x)}=\frac{F''(x)}{F(x)}-2=\lambda \quad \text{KONSTANT}$$

$\cdot)$ $\dfrac{F''(x)}{F(x)}=\lambda+2 \Rightarrow (\lambda+2)F(x)=F''(x)$

$\cdot)$ $\lambda+2=0 \Rightarrow F''(x)=0 \Rightarrow F(x)=Ax+B$

$$\left.\begin{aligned} F(0)=0 &\Rightarrow B=0 \\ F'(\tfrac{\pi}{2})=0 &\Rightarrow A=0 \end{aligned}\right\} \Rightarrow u\equiv 0 \ \text{(UØNSKET)}$$

$\cdot)$ $(\lambda+2)>0 \Rightarrow F(x)=Ae^{\sqrt{2+\lambda}\,x}+Be^{-\sqrt{2+\lambda}\,x}$

$$\left.\begin{aligned} F(0)=0 &\Rightarrow A+B=0 \\ F'(\tfrac{\pi}{2})=0 &\Rightarrow A\left(e^{\sqrt{2+\lambda}\frac{\pi}{2}}+e^{-\sqrt{2+\lambda}\frac{\pi}{2}}\right)=0 \end{aligned}\right\} \Rightarrow F\equiv 0 \Rightarrow u\equiv 0 \ \text{(UØNSKET)}$$

$\cdot)$ $(\lambda+2)<0 \Rightarrow F(x)=A\cos(\sqrt{-(\lambda+2)}\,x)+B\sin(\sqrt{-(\lambda+2)}\,x)$

$F(0)=0 \Rightarrow A=0$

$$F'(\tfrac{\pi}{2}) \Rightarrow B\cos(\sqrt{-(\lambda+2)}\,\tfrac{\pi}{2})=0 \Rightarrow \begin{cases} B=0 \Rightarrow u\equiv 0 \ \text{(UØNSKET)} \\ B\neq 0 \text{ OG } \sqrt{-(\lambda+2)}=(2n+1) \quad n=0,1,2 \end{cases}$$

$$\Rightarrow \lambda+2=-(2n+1)^2$$
$$\lambda=-(2n+1)^2-2$$

$\Rightarrow$ LET $F_n(x)=B_n\sin((2n+1)x)$

$\Rightarrow$ $G_n(t)$ OPPFYLLER $\quad \dfrac{G_n'(t)}{(1+t^2)G_n(t)}=\lambda=-(2n+1)^2-2$

$$\Rightarrow G_n(t)=G_n(0)e^{\lambda(t+\frac{t^3}{3})}$$

LØSNINGER PÅ FORMEN

$$u(x,t)=\sum_{n=0}^{\infty}B_n\sin((2n+1)x)e^{-((2n+1)^2+2)(t+\frac{t^3}{3})}$$

b) FINN EN LØSNING SOM TILFREDSSTILLER (A) OG (B) OG I TILLEGG RANDBETINGELSEN

$$u(x,0) = \sin(3x) + \sin(17x) \qquad x \in [0, \tfrac{\pi}{2}]$$

$$u(x,0) = \sum_{n=0}^{\infty} B_n \sin((2n+1)x) \Rightarrow \begin{array}{l} B_1 = 1 \\ B_8 = 1 \\ \text{ELLERS } B_n = 0 \end{array}$$

$$\Rightarrow u(x,t) = \sin(3x) e^{-9(t+\frac{t^3}{3})} + \sin(17x) e^{-289(t+\frac{t^3}{3})}$$